Capital asset pricing model with fuzzy returns and hypothesis testing

Alfred Mbaïradjiim M. Lameta
Université Montpellier I

J. Sadefo Kamdem * Lameta
Université Montpellier I

A. Shapiro
Smeal College of Business
Penn State University

M. Terraza Lameta-CNRS
Université Montpellier I

First draft

Abstract

Over the last four decades, several estimation issues of the beta have been discussed extensively in a large literature. An emerging consensus is that the betas are time-varying and their estimates are impacted upon the return interval and the length of the estimation period. These findings lead to the prominence of the practical implementation of the Capital Asset Pricing Model. Our goal in this paper is two-fold: After studying the impact of the return interval on the beta estimates, we analyze the sample size effects on the preceding estimation. Working in the framework of fuzzy set theory, we first associate the returns based on closing prices with the intraperiod volatility for the representation by the means of a fuzzy random variable in order to incorporate the effect of the interval period over which the returns are measured in the analysis. Next, we use these fuzzy returns to estimate the beta via fuzzy least square method in order to deal efficiently with outliers in returns, often caused by structural breaks and regime switches in the asset prices. A bootstrap test and an asymptotic test are carried out to determine whether there is a linear relationship between the market portfolio fuzzy return and the given asset fuzzy return. Finally, the empirical results on French stocks reveal that our beta estimates seem to be more stable than the ordinary least square (OLS) estimates when the return intervals and the sample size change.

Key-words: CAPM ; Beta; Fuzzy least squares; Bootstrap hypothesis testing; Intraperiod volatility.

*Corresponding author. LAMETA Université de Montpellier I, UFR d’Economie Avenue Raymond DUGRAND - Site de Richter C.S. 79606 34960 MONTPELLIER CEDEX 2 France, Courriel: sadefo@lameta.univ-montp1.fr, jules.sadefo-kamdem@univ-montp1.fr
1 Introduction

By introducing the Capital Asset Pricing Model (CAPM), Sharpe (1963), Lintner (1965a, b) proposed the first quantified approach of risk in quantitative finance. They determined the systematic risk or beta by covariance with the market. The beta is commonly computed by applying the standard market model estimated under ordinary least square (OLS).

As with most important scientific models, the CAPM has been subject to substantial criticism, the most famous of which addressed by Fama and French (1992). A large literature over several decades focused on the estimation issue of the CAPM, more precisely, of the systematic risk or beta. The stability of the beta across time was discussed by some authors such as Blume (1971, 1975), Ferson and Harvey (1991) and some references therein. The time-varying nature of the betas was the common finding of these studies. This conclusion motivated the introduction of some conditional versions of beta and a formal characterization of their persistence and predictability vis-à-vis their underlying components by Andersen et al. (2004). Another important estimation issue of the beta, extensively investigated by several authors, is the impact that the return interval has on the beta estimate (for details, see Brailsford and Josev (1997), Gençay et al. (2005), Hawawini (1983) and some references therein). These studies report that different values of the beta can be estimated by changing the return interval over the same sample period. Furthermore, Handa et al. (1993) empirically reject the CAPM when monthly returns are used but they accept the CAPM with yearly returns whereas Fama (1981, 1990) show that the power of macroeconomic variables in explaining the stock prices increased with time length. However, Levhari and Levy (1977) suggest to use the relevant time horizon (implicit) in the decision making process of investors in order to avoid biasing the beta estimate even if information is available in a higher frequency.

In this paper, we study the CAPM which remains central to financial economics despite the numerous criticisms that it is subject to. We propose a new approach for estimating the systematic risk. The motivation behind our approach comes from two sources. First, we want to use a relevant return interval in the investor’s decision making process and to take into account the higher frequency data observed within the considered interval, in order to decrease the loss of information often caused by a huge discretization in time. Second, we aim to introduce an estimation method which is able to deal better with the impacts of the estimation period length (Kim (1993)), and that structural breaks and other regime switches (Bundt et al.(1992), Garcia and Ghysels (1998)) on the betas estimate. The basic assumption of our modeling approach is the representation of financial assets returns through a fuzzy random variable in order to capture different relevant information on the probability distribution of the intraperiod returns observed. The fuzzy representation of financial asset return has been realized in literature by many authors including among others, Tanaka and Guo (1999), Parra et al. (2001), Terol et al. (2006) et Yoshida (2009).

\footnote{A conceptual discussion on fuzzy random variable is given by Shapiro (2009)}
We study the fuzziness of returns over a period, as the intraperiod volatility. This modeling aims to deal with the impact of return intervals on the returns representation and to produce estimates less sensitive to sample size effects. We apply fuzzy computation method to define the fuzzy returns and to fit the fuzzy market line (FML) as a fuzzy linear regression model. Our fuzzy set-valued returns can be seen as a generalized form of the interval-valued one. Furthermore, the fuzzy linear regression model introduced by Tanaka et al. (1982) and refined by Diamond (1988), is more adequate to dealing with outliers, and small sample size as it is in our case, than the linear regression model.

The remainder of this paper is organized as follows. The section 2 expose the fuzzy presentation process of monthly returns. Section 3 is an overview of the existing literature on the beta estimation issues. Section 4 introduces the fuzzy market line, its implications and the hypothesis testing approaches of the beta estimates. Section 5 is an empirical study based on the French index which compares our beta estimates to the OLS estimates and the wavelet estimates. Finally some conclusions are listed in Section 6. In addition for better understanding of the paper, a brief presentation of basic concepts of fuzzy set theory is given in Appendix A.

2 Fuzzy representation of returns
Fuzzy random variables (FRV) were introduced and defined by Kwakarnak (1978), Puri and Ralescu (1986) as a well-formalized model for fuzzy set-valued random elements. Since these definitions, numerous studies in probability theory have been developed to analyze the properties of this new class of random variables. For the last three decades, we can cite those related to the formalization of the measurability, to the laws of large numbers which strengthen the suitability of the fuzzy mean and to the hypothesis testing (See Gil et al. (2006) for an overview of these developments). Despite the existence of this complete mathematical analysis framework, the application of these theoretical results is still quite limited because of the difficulties that we have met to observe and measure FRVs in practice. Hence the necessity of building methods to provide fuzzy representations of observations, which are often crisp. A solution was proposed by Gonzalez-Rodriguez et al. (2006) who introduced a family of fuzzy representation of random variables. Each of the representations transforms a crisp random variable into a fuzzy random variable whose mean captures different relevant information on the probabilistic distribution of the original real-valued random variable. However the application of this method requires à priori assumptions about the distribution of the real variable and about the shape of the membership function of the fuzzy random variable. This double assumption may lead to a significant bias of information.

In this section, we propose to give a fuzzy representation of a financial asset return over one month by using past observations at higher frequency (daily returns). We will give a parametric
representation of the monthly return through a fuzzy set by making an assumption on the left and right shape functions.

The asset price time series is initially partitioned into sub-groups according to periods (months in our case). For each month, daily returns successively observed are calculated and we consider their corresponding empirical probabilistic distribution. Then the first two moments (expected value and variance) and the return over the month, based on the first and last values of the price time series are computed. The monthly return is then represented as a symmetric LR-fuzzy number with a central value equal to the return over the month and the spread is the scaled standard deviation.

The fuzzy representation process can be summarized as in the following procedure.

**Procedure 2.1**

1. **Step 1:** Partition price time series in sub-groups \( P_i = \{P_{i1}^1, ..., P_{in}^i\} \) with size \( n \) each one corresponding to a period \( i \). The size sample \( n \geq 2 \) has to be sufficiently large.

   For each period \( i \)

2. **Step 2:** Compute the return over the period \( R_i = \frac{P_{in}^i - P_{i1}^i}{P_{i1}^i} \)

3. **Step 3:** Compute the returns within the period \( R_{ij}^i = \frac{P_{ij+1}^i - P_{ij}^i}{P_{ij}^i} \), \( j \in \{1, ..., n - 1\} \)

4. **Step 4:** Estimate empirically the variance \( \hat{\sigma}_i^2 \) of \( R_{ij}^i \) as

   \[
   \hat{\sigma}_i^2 = \frac{1}{n-1} \sum_{j=1}^{n-1} (R_{ij}^i - \hat{\mu}_i)^2
   \]

   where

   \[
   \hat{\mu}_i = \frac{1}{n-1} \sum_{j=1}^{n-1} R_{ij}^i
   \]

5. **Step 5:** Scale the intraperiod volatility by \( \Delta_i = \sqrt{n-1} \hat{\sigma}_i \)

6. **Step 6:** Fit the membership function of the symmetric LR-fuzzy return with central value \( R_i \) and spread \( \Delta_i \)

The spread of the fuzzy returns is hence defined as the periodic volatility computed using standard scaling practise which consists in multiplying the realized volatility and square root of the number of returns observations within the period. This scaling method also known as the

---

\(^2\)In this study, we use daily closing prices time series which are partitioned into monthly periods hence \( n = 20 \)

\(^3\)A discussion on this scaling method is given by Danielsson and Zigrand (2005)
square-root-of-time rule, requires the assumption that the returns are independent and identically distributed. The fuzzy return then allows us to associate the closing prices-based returns to some other relevant information relative to their probabilistic distributions.

Finally, we have the following statement

**Proposition 1** If returns successively observed are supposed to be real random variables, the symmetric LR-fuzzy set constructed by the Procedure 2.1 is a fuzzy random variable as described in the definition A.3.

3 Prior research

Beta is typically defined as the slope parameter of the Sharpe’s market line which is given by

\[ R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it}, \]

where \( R_{it} \) and \( R_{mt} \) are respectively the realized returns on the asset \( i \) and on the market portfolio \( m \) over the return interval \( t \); \( \alpha_i \) is the constant term for the asset \( i \); the error term \( \epsilon_{it} \) is a gaussian random variable defined through the variance (\( \text{Var} \)) and the covariance (\( \text{Cov} \)) as follows

- zero-mean: \( \mathbb{E}(\epsilon_{it}) = 0, \forall t, \forall i \)
- homoscedastic: \( \text{Var}(\epsilon_{it}) = \sigma_{\epsilon_i} \)
- mutually uncorrelated in the time: \( \text{Cov}(\epsilon_{it}, \epsilon_{it'}) = 0, \forall t \neq t' \)
- \( \epsilon_{it} \) are independent to market return \( R_{mt} \): \( \mathbb{E}(\epsilon_{it} | R_{mt}) = 0, \forall t, i \).

\( \beta_i \) is the sensitivity of the asset returns \( i \) to the market portfolio returns. It is assumed to be constant and it is practically estimated over a finite number of return measurement intervals using ordinary least square (OLS). The assumption of the of beta’s time-constancy and its estimation method lead to some limitations.

One of the first criticisms is the instability of its estimates across time. Some researchers such as Blume (1971, 1975), Ferson and Harvey (1991, 1993) and Ferson and Korajczyk (1995) find that estimated betas exhibit statistically significant time variation. For this reason, these last three works suggest to replace the static CAPM by some forms of time-varying beta and of conditional CAPM. This approach allows out-performing the constant beta if the dynamics of the beta is captured with success. However, as shown by Ghysels (1998), if the beta risk is misspecified, there is a real possibility to commit pricing errors potentially larger than with a constant beta assumption.

The impact that the return interval has on the beta estimate is another important issue largely discussed in the literature. Brailsford and Josev (1997), Hawawini (1983), Handa et al. (1989, 1993) etc; report that different beta estimates can be obtained over the same period by
changing the interval over which the return is calculated. Moreover, it is shown by Fama (1981, 1990) that the power of macroeconomic variables in explaining the stock prices increased with increasing time length whereas the early work of Levhari and Levy (1977) provides evidence that the beta estimates were biased if the analyst used a shorter time horizon than the relevant time horizon implicit in the decision making process of investors. These two conclusions imply the necessity to find a tradeoff between the adequate time horizon and the frequency of the available information. A solution based on wavelet analysis, was proposed by Gençay et al. (2005). They decomposed the time series, measured at the highest possible frequency into different time scales before investigating the beta behavior at different time horizons without losing information.

Financial assets closing prices are generally noised because of the imperfections of the trading process. As noticed in Aït-Sahalia et al. (2010), these imperfections might be largely divided into three points. The first represents the frictions inherent in the trading process: bid-ask bounces, discreteness of price changes and rounding, trades occurring on different markets or networks, etc. The second point concerns informational effects such as differences in trade sizes or informational content of price changes. The last point encompasses measurement or data recording errors. Then returns based on closing prices are variables measured with errors. It follows that the use of returns based on closing prices, tends to lead to inconsistent ordinary least square estimators in market line or multifactor models \(^4\). Cragg (1994) demonstrated that the slope coefficients are biased toward zero and concluded that the measurement error "produces a bias of the opposite sign on the intercept coefficient when the average of the explanatory variables is positive".

4 Klepper and Leamer (1984), Leamer (1987) among others provide evidence of inconsistency of ordinary least square estimators in linear regression models with measurement errors in the regressors.

### 4 The fuzzy market line (FML)

The numerous estimation issues of the systematic risk by OLS method listed in the previous section motivate to introduce in the following a new estimation method of the beta. Our aim is to analyze the effect of the return interval and the impact of estimation period length on the beta estimates. The fuzzy representation of a period incorporates the interval over which the return is calculated by the mean of the scaled volatility. Regimes switches or structural breaks in the price process imply likely an outlier in the return time series. When the sample size increases by taking into consideration this kind of observation, the beta estimate by OLS method change especially when there is a too small data set. In order to deal efficiently with these limitations, we propose to define the one-factor market model as a fuzzy linear regression model. The slope parameter of this model defines our systematic risk estimate.
4.1 Assumptions and model specification

Assumptions of CAPM with fuzzy returns include two aspects: economy and preference of investors.

We adopt the assumption of securities market economy commonly used in asset pricing theory, in which all investors hold their wealth in the form of financial assets. We consider a single-period securities market economy and we assume that there exist one riskless asset and \( n \) risky assets in time \( t \). Except that assumption of normal distribution of return is replaced by the assumption the returns are LR-fuzzy random variables, other assumptions are the same as in the original CAPM, including that capital market is completely competitive and frictionless, capital market clearing, riskless borrowing, and lending are allowed. The LR-fuzzy random return vector is denoted \( \tilde{\mathbf{R}} = (\tilde{R}_1, ..., \tilde{R}_n) \) and the riskless return is \( R_f \). Since \( R_f \) is not subjected to uncertainty and known with precision, it is defined as real number instead of a fuzzy number.

For the preference of investors, we assume that each investor cares only about the Auman fuzzy expected return \( \mathbb{E}[\tilde{R}_i^A] \) and its underlying standard deviation \( \sigma_i^A \) and all investors have the same beliefs about investment opportunities: \( \mathbb{E}[\tilde{R}_i^A], \sigma_i^A \) and all correlations \( \sigma_{ij}^A \) for the \( n \) risky assets \((i, j = 1, ..., n)\).

Under these assumptions, if we note \( \tilde{R}_m \) the fuzzy return of the market portfolio, we mean by fuzzy market line\(^5\) the fuzzy linear regression

\[
\tilde{R}_i = \tilde{\alpha}_i + \beta \tilde{R}_m + \tilde{\epsilon}_i, \quad \tilde{\alpha}_i \in \mathcal{F}(\mathbb{R}), \quad \beta \in \mathbb{R}.
\]  

(2)

Where \( \tilde{\epsilon}_i \) is a LR-fuzzy random variable, specified as follows:

\[
\mathbb{E}^A(\tilde{\epsilon}_i) = \mathbf{1}_{\{0\}}, \quad \forall i = 1, ..., n;
\]

\[
\text{Cov}^A(\tilde{\epsilon}_i, R_m) = 0, \quad \forall i = 1, ..., n;
\]

\[
\text{Cov}^A(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, \quad \forall i, j = 1, ..., n.
\]

\( \mathbf{1}_{\{0\}} \) stands for the indicator function of a classical set.

The crisp parameter \( \beta_i \) is the sensitivity of the fuzzy return of the asset \( i \) to the market portfolio fuzzy returns as in the standard market line. It represents the systematic risk under the above-mentioned assumptions. If the spreads of the fuzzy returns are null, then it degenerates to the slope parameter of the standard market line (of the CAPM). The intercept (fuzzy number-valued) parameter \( \tilde{\alpha}_i \) can be interpreted as the fuzzy Jensen’s alpha.

\(^5\)The specification \( \tilde{R}_i = \tilde{\alpha}_i + \beta \tilde{R}_m + \tilde{\epsilon}_i \) for the fuzzy market is also possible. However, the product \( \beta \tilde{R}_m \) of two LR-fuzzy numbers does not always provide a LR-fuzzy number (Oussalah and De Schutter (2003)) and consequently implies further difficulty in the implementation of the fuzzy least square method for the model estimation. We limit the present study to the crisp beta and an extension to a fuzzy beta needs to be addressed in future research.
4.2 Model estimation

The estimation problem of the linear regression model with fuzzy data has been previously treated in the literature from different point of views. Näther (2006) gives a large overview of the main approaches. In this paper, we opt for the Wünsche and Näther (2002)’s approach which analyzes the problem in the particular case of convex fuzzy random variables on $\mathbb{R}^n$. Using the $\delta_2$ metric presented in equation 16, they reduce the estimation to the following optimization problem

$$\inf_{\tilde{\alpha}_i \in \mathcal{F}({\mathbb{R}}), \beta \in \mathbb{R}} \mathbb{E}(\delta_2^2(\tilde{R}_i, \tilde{\alpha}_i + \beta \tilde{R}_m))$$

(3)

The solution of this least square problem is given by

Theorem 4.1 (Wünsche and Näther (2002))

Let $\tilde{R}_i$ and $\tilde{R}_m$ be two square-integrable fuzzy random variables and consider the optimization problem

1. Let be $\beta \geq 0$: If $\text{Cov}(\tilde{R}_i, \tilde{R}_m) \geq 0$ and $\mathbb{E}^A(\tilde{R}_i) \ominus_H \text{Cov}(\tilde{R}_i, \tilde{R}_m) \text{Var}(\tilde{R}_m) \mathbb{E}^A(\tilde{R}_m)$ exists then

$$\beta^* = \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)}, \quad \tilde{\alpha}_i^* = \mathbb{E}^A(\tilde{R}_i) \ominus_H \beta^* \mathbb{E}^A(\tilde{R}_m)$$

2. Let be $\beta \leq 0$: If $\text{Cov}(\tilde{R}_i, -\tilde{R}_m) \geq 0$ and $\mathbb{E}^A(\tilde{R}_i) \ominus_H -\frac{\text{Cov}(\tilde{R}_i, -\tilde{R}_m)}{\text{Var}(\tilde{R}_m)} \mathbb{E}^A(\tilde{R}_m)$ exists then

$$\beta^* = \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)}, \quad \tilde{\alpha}_i^* = \mathbb{E}^A(\tilde{R}_i) \ominus_H \beta^* \mathbb{E}^A(\tilde{R}_m)$$

are solutions of (3)

In the special case of LR-fuzzy returns, the beta above-estimated can be relied to the beta of the classical CAPM as follows

Proposition 4.2 If $\beta_0$ is the systematic risk estimated from the Sharpe market and $\beta$ the systematic risk from the fuzzy market line, then there exists a unique strictly positive real number $\lambda$ such as

$$\beta = \beta_0 + \lambda [\text{cov}(\Delta_i, \Delta_m) \text{var}(R_m) - \text{cov}(R_i, R_m) \text{var}(\Delta)]$$

(4)

Before interpreting this relationship, we first define a quantity $\beta_\Delta = \frac{\text{cov}(\Delta_i, \Delta_m)}{\text{Var}(\Delta_m)}$. $\beta_\Delta$ determines the sensitivity of the asset intraperiod volatility (spread of fuzzy return) to the market portfolio one. Then, the Proposition 4.2 implies that

Corollary 4.3

$$\beta > \beta_0 \iff \beta_\Delta > \beta_0$$

(5)
The equivalence (5) can be interpreted as follows: If the sensitivity of the asset intraperiod volatility (spread of fuzzy return) to the market portfolio one is higher than the systematic risk estimate based on the considered period return (the standard beta), then the beta of the fuzzy market is an upward correction of the systematic risk. If there is not some linear relationship between the intraperiod volatility of the asset and the intraperiod market portfolio volatility, then our beta estimate degenerates to the classical one. This interpretation emphasizes the fact that the beta of the fuzzy market line appears as a generalization of the standard beta resulting from the incorporation of the intraperiod volatility in the return representation of an asset for a given period.

4.3 Hypothesis testing on beta estimate

The goal of this subsection is concluding on whether or not the systematic risk beta, introduced by the FML, vanishes. In other words, we want to assess the suitability of the assumption that the fuzzy returns of the considered asset is linearly related to market portfolio fuzzy return by testing the hypothesis of the nullity of the slope parameter of the fuzzy linear model. The subsection is largely attributable to Gil et al. (2007).

The hypotheses are formulated as follow

\[ \begin{align*}
    H_0 : & \beta_i = 0 \\
    H_1 : & \beta_i \neq 0
\end{align*} \] (6)

We have the following equivalent hypotheses

Lemma 4.4 If \( \text{Cov}(R_i, R_m) \) and \( \text{Cov}(\Delta_i, \Delta_m) \) have the same sign and if \( R_m \) it is not almost surely constant, then the hypotheses (6) are equivalent to the following assertions

\[ \begin{align*}
    H_0 : & \text{Cov}(R_i, R_m)^2 + \text{Cov}(\Delta_i, \Delta_m)^2 = 0 \\
    H_1 : & \text{Cov}(R_i, R_m)^2 + \text{Cov}(\Delta_i, \Delta_m)^2 > 0
\end{align*} \] (7)

Then, the Lemma 4.4 states that, testing \( H_0 \) against \( H_1 \) reduces to test the equivalent hypotheses about the covariance matrix of the classical random vector \( \mathbf{R} = (R_m, R_i, \Delta_m, \Delta_i) \) on the basis of a random sample of \( n \) independent random vectors \( \mathbf{R}_1, \ldots, \mathbf{R}_n \) with \( \mathbf{R}_j = (R_{jm}, R_{ji}, \Delta_{jm}, \Delta_{ji}) \) for \( j = 1, \ldots, n \).

Gil et al. (2007) developed an exact and an asymptotic method for testing the above-specified hypotheses. The exact method firstly introduced appears to be unrealistic and in some sense limited for real-life applications because of the difficulty to find a well-supported model for the distributions of interval-valued random sets. For these reasons, next, we opt for the asymptotic and the bootstrap approaches to test our hypotheses.
We assume that available samples are large and that the population is finite. Moreover, if we assume that not all the fuzzy returns are nondegenerate (i.e. random), by taking advantage of the large sample theory, an asymptotic approach to test $H_0$ versus $H_1$ is given by

**Theorem 4.5 (Gil et al. (2007))**

If $R_j = (R_{mj}, R_{ij})$ and $\Delta_j = (\Delta_{mj}, \Delta_{ij})$ are independent random vectors for each $j \in \{1, ..., n\}$, then, to test at the nominal significance level $\gamma$ the null hypothesis $H_0 : a = 0$, against the alternative $H_1 : a \neq 0$,

1. if the following conditions:
   \[
   [R_m - E(R_m)][R_i - E(R_i)] \text{ is nondegenerate,} \tag{8}
   \]
   \[
   [\Delta_m - E(\Delta_m)][R_i - E(\Delta_i)] \text{ is nondegenerate} \tag{9}
   \]
   are satisfied, the null hypothesis $H_0$ should be rejected if
   \[
   n \left\{ \frac{[Cov(R_m, R_m)]^2}{Var([R_m - E(R_m)][R_i - E(R_i)])} + \frac{[Cov(R_i, R_i)]^2}{Var([\Delta_m - E(\Delta_m)][R_i - E(\Delta_i)])} \right\} > \chi^2_{2, \gamma},
   \]
   where $\chi^2_{2, \gamma}$ is the $100(1 - \gamma)$ fractile of the chi-square distribution degrees of freedom.

2. If condition (8) holds but condition (9) is not satisfied, the null hypothesis $H_0$ should be rejected if
   \[
   n[Cov(R_m, R_m)]^2 / Var([R_m - E(R_m)][R_i - E(R_i)]) > \chi^2_{1, \gamma},
   \]

3. If condition (9) does not hold but condition (8) is satisfied, the null hypothesis $H_0$ should be rejected if
   \[
   n[Cov(\Delta_m, \Delta_m)]^2 / Var([\Delta_m - E(\Delta_m)][R_i - E(\Delta_i)]) > \chi^2_{1, \gamma}.
   \]

Moreover, in the three cases the probability of rejecting $H_0$ under the alternative hypothesis $H_1$ converge to 1 as $n \to +\infty$

When the sample size is insufficient for straightforward statistical inference, the application of the bootstrap approach will become even more valuable than the one for random variables or vectors which requires assumptions about the probabilistic distributions. The bootstrap techniques with interval data have been applied in the literature including, among others, Efron (1981) and Hall et al. (2001).

Beran and Srivastava (1985) have designed a valuable bootstrap method for dealing with random vectors. The method of Beran and Srivastava can then be applied to test $H_0$, more
precisely the nullity of the four elements $\sigma_{12}, \sigma_{34}, \sigma_{21}$ and $\sigma_{43}$ of the covariance matrix $\Sigma = (\sigma_{ij})_{ij}$ (of order $4 \times 4$) of the random vector $(R_m, R_i, \Delta_m, \Delta_i)$ defined by

$$
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
$$

with

$$
\Sigma_{11} = \begin{pmatrix}
\text{Var}(R_m) & \text{Cov}(R_m, R_i) \\
\text{Cov}(R_m, R_i) & \text{Var}(R_i)
\end{pmatrix}
$$

$$
\Sigma_{12} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m) & \text{Cov}(R_m, \Delta_i) \\
\text{Cov}(\Delta_m, R_i) & \text{Cov}(R_i, \Delta_i)
\end{pmatrix}
$$

$$
\Sigma_{21} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m) & \text{Cov}(R_i, \Delta_m) \\
\text{Cov}(\Delta_m, R_i) & \text{Cov}(R_i, \Delta_i)
\end{pmatrix}
$$

$$
\Sigma_{22} = \begin{pmatrix}
\text{Var}(\Delta_m) & \text{Cov}(\Delta_m, \Delta_i) \\
\text{Cov}(\Delta_m, \Delta_i) & \text{Var}(\Delta_i)
\end{pmatrix}
$$

The particularization of the method by Beran and Srivastava (1985) is presented by Gil et al. (2007) as follows

**Algorithm 4.6**

**Step 1**: Compute the variance-covariance matrix of $Z_j = (R_{mj}, R_{ij}, \Delta_{mj}, \Delta_{ij})$ for $j = 1, \ldots, n$

$$
\hat{S} = \begin{pmatrix}
\hat{S}_{11} & \hat{S}_{12} \\
\hat{S}_{21} & \hat{S}_{22}
\end{pmatrix}
$$

where

$$
\hat{S}_{11} = \begin{pmatrix}
\text{Var}(R_m) & \text{Cov}(R_m, R_i) \\
\text{Cov}(R_m, R_i) & \text{Var}(R_i)
\end{pmatrix}
$$

$$
\hat{S}_{12} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m) & \text{Cov}(R_m, \Delta_i) \\
\text{Cov}(\Delta_m, R_i) & \text{Cov}(R_i, \Delta_i)
\end{pmatrix}
$$

$$
\hat{S}_{21} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m) & \text{Cov}(R_i, \Delta_m) \\
\text{Cov}(\Delta_m, R_i) & \text{Cov}(R_i, \Delta_i)
\end{pmatrix}
$$

$$
\hat{S}_{22} = \begin{pmatrix}
\text{Var}(\Delta_m) & \text{Cov}(\Delta_m, \Delta_i) \\
\text{Cov}(\Delta_m, \Delta_i) & \text{Var}(\Delta_i)
\end{pmatrix}
$$

and the value of statistic $\Theta = nh(\hat{S})$, where

$$
h(\hat{S}) = \text{Cov}(R_m, R_i)^2 + \text{Cov}(\Delta_m, \Delta_i)^2
$$
**Step 2**: compute

$$\pi(\hat{S}) = \begin{pmatrix} \pi_{11}(\hat{S}_{11}) & \hat{S}_{12} \\ \hat{S}_{21} & \pi_{11}(\hat{S}_{22}) \end{pmatrix}$$

where

$$\pi_{11}(\hat{S}_{11}) = \begin{pmatrix} \text{Var}(R_m) & 0 \\ 0 & \text{Var}(R_i) \end{pmatrix}$$

$$\pi_{22}(\hat{S}_{22}) = \begin{pmatrix} \text{Var}(\Delta_m) & 0 \\ 0 & \text{Var}(\Delta_i) \end{pmatrix}$$

**Step 3**: Compute the bootstrap population

$$V_j = (Z_i \hat{S}^{-1/2} \pi(\hat{S})^{1/2}) = (V_j^1, V_j^2, V_j^3, V_j^4), \quad \forall j = 1, \ldots, n$$

Note that the bootstrap population \((V_1, \ldots, V_n)\) is defined with the aim of guaranteeing that the sample information of the variance-covariance matrix is preserved as much as possible, as well as the null hypothesis \(H_0\) is fulfilled.

**Step 4**: Obtain a sample of independent and identically distributed random vectors \((V_1^*, \ldots, V_n^*)\) from the bootstrap population.

**Step 5**: Compute the value of bootstrap statistic

$$\theta^* = n[\text{Cov}(V^{1*}, V^{2*})^2 + \text{cov}(V^{3*}, V^{4*})^2]$$

**Step 6** Steps 4 and 5 should be repeated a large number \(B\) of times to get a set of \(B\) estimators, denoted by \(\{\theta_1^*, \ldots, \theta_B^*\}\).

**Step 7** Compute the bootstrap \(p\)-value as proportion of values in \(\{\theta_1^*, \ldots, \theta_B^*\}\) being greater than \(\theta\).

Finally, the decision rule is given by

**Theorem 4.7** (Gil et al.(2007)) To test at nominal significance level \(\gamma\) the null hypothesis 
\(H_0 : \beta = 0\) against alternative \(H_1 : \beta \neq 0\), \(H_0\) should be rejected if

$$\theta = nh(\hat{S}) > c_\gamma$$

where \(c_\gamma\) is the \(100(1 - \gamma)\) fractile of the distribution of \(\theta^*\) and \(n\) the number of observations.

### 5 Empirical studies

In this section, we empirically investigate the FML-estimated systematic risk estimate. We first analyze whether there is some linear relationship between the fuzzy return of the considered asset and the fuzzy return of the market portfolio by testing statistically the nullity of the beta estimate. Next, we assess the impacts of the interval return and of the sample size length on
our beta estimates.

The main objective of the section is to investigate empirically the properties of FLS beta estimates. We compare over our dataset, the behaviors of the FLS beta, of the OLS beta and of the beta estimated by the Gençay et al. (2005) wavelet multiscaling method when the return interval and the sample size change.

Our data set available on Yahoo Finance France, consists of 30 stocks quoted in the French financial market of Paris and the CAC40 index\(^6\) is taken as the market portfolio. The sample period including the late-2000s financial crisis, covers April 2007 to December 2009. The choice of this relatively short estimation period is motivated by our wish to use only recent observations in order to reflect current information in our estimate. The table lists the stocks names, their tickers used in this paper and statistics summary of monthly returns based on closing prices without including dividends.

Weakly, biweekly, triweekly and monthly fuzzy returns are successively constructed, by associating crisp return based on closing prices to intraperiodic volatility following the Procedure presented in section. We assume that the fuzzy returns have triangular membership functions. Extensions to other membership functions needs to be addressed in future work. For each return interval, the systematic risks are estimated and the Figure compares the fuzzy least square (FLS), the ordinary least square (OLS) estimates and the wavelet estimates at different return intervals and scales. Before studying the behavior of our beta estimate when the return interval and the estimation period length change, we test the null hypothesis that it vanishes.

For the asymptotic approach, the test statics computed under the assumption of non-degeneracy of the all used random variables (returns), are reported in the Table. These statics are superior to the theoretical 99% fractile of the chi-square distribution with 2 degrees of freedom (equal to 9.21). Hence the rejection of the null hypothesis \(H_0\) at the nominal significance level 1%.

For the bootstrap approach, we apply the Algorithm to compute the bootstrap p-value of the statistic \(\theta^*\) and 1000 bootstrap replications have been considered. The Table listing the test statistic and the 99% fractile of the distribution of \(\theta^*\), shows that the null hypothesis \(H_0\) is rejected at the nominal significance level 1% for all considered stocks, whence the relevance of the linear relationship assumed by the fuzzy one-factor market model and the suitability of our beta estimate.

\(^6\)The CAC40 index is a capitalization-weighted measure of the 40 highest market caps on the Paris Stock Exchange (Euronext).
The above-realized statistical hypothesis testing provides the clearest evidence of the linear relationship assumed by the fuzzy one-factor market model and the suitability of our beta estimate. Next, in a comparative analysis, we will assess the behavior of the beta when the return interval or the length of the estimation period change. For the interval effect, the properties of the wavelet multiscaling beta, the OLS beta and the FLS beta will be compared. Since the wavelet multiscaling betas are computed using the same return interval for different period dynamics, only the OLS beta and the FLS beta will be evaluated for the impact of the training sample size.

Gencay et al. (2005) introduced the multiscale systematic risk to overcome the interval effect in the beta estimation. Their estimation method is based on a wavelet analysis that enables to decompose a time series into different time scales. For a given asset $m$, they first apply the wavelet transform to the return time series $r_{mt}$ yielding the wavelet coefficient vector $w_m$.

By assuming that dependence structure of the return $r_{mt}$ is independent of time, the they
Table 2: This table shows that the computed test statistics are superior to the theoretical 99% fractile of the chi-square distribution with 2 degrees of freedom the whence the rejection of the null hypothesis $H_0$ at the nominal significance level 1% all stocks.

Define the time-independent wavelet variance of asset $m$ with level $j$ to be

$$\sigma^2_{mj} = \text{Var}(\hat{w}_{mj})$$

Let $r_{mt}$ and $r_{nt}$ be the return for two distinct assets $m$ and $n$. Similarly, the wavelet covariance between $r_{mt}$ and $r_{nt}$ for level $j$

$$\sigma_{nmj} = \text{Cov}(\hat{w}_{mj}, \hat{w}_{nj}).$$

Finally the wavelet beta of asset $n$ at level $j$ is estimated as follows

$$\beta^w_{nj} = \frac{\text{Cov}(\hat{w}_{mj}, \hat{w}_{nj})}{\text{Var}(\hat{w}_{mj})}.$$
Table 3: This table emphasizes that \( \theta \) statistic is strictly superior to the 99\% fractile of the distribution of \( \theta^* \) whence the rejection of the null hypothesis \( H_0 \) at the nominal significance level 1\% for every considered stocks.

For more details on the wavelet variance and covariance, confer Gençay et al. (2001a, Ch. 7) and references therein.

For the practical estimation of the multiscale systematic risk, we employ the daily return time series. We use the Daubechies least asymmetric wavelet filter of length 8 to transform these time series. We estimate the wavelet beta for scales \( j = 1, ..., 4 \) such that scale 1 is associated with 2 – 4 day dynamics, scale 2 is associated with 4 – 8 day dynamics, scale 3 is associated with 8 – 16 day dynamics and scale 4 is associated with 16 – 32 day dynamics. In the comparative analysis, we suppose that scale 1 corresponds to weekly return interval, scale 2 corresponds to biweekly return interval, scale 3 corresponds to triweekly return interval and scale 4 corresponds to monthly return interval.

The Figure[1] shows that different values of the beta can be estimated by changing the interval return over the same sample period as well as by OLS, FLS and wavelet. And the Figure
indicates that the OLS and FLS estimates are generally very close while the wavelet beta is relatively farther for all return intervals. However, as emphasized by the Figure which depicts the coefficient of variation of betas for different interval returns, the FLS estimate of beta has a smaller value of the coefficient of variation over the sample of the studied stocks, than the OLS and the wavelet beta estimates. The mean values, over all stocks and return intervals, of the coefficient of variation are 9.67% for FLS betas, 10.96% OLS betas and 13.01% in our study sample hence the relative improvements reported on the Table. In conclusion, the use of FLS estimation method allows reducing the effect of the interval return on the systematic risks estimation for the stocks used in our study.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Improvement rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>FLS over OLS</td>
<td>11.77%</td>
</tr>
<tr>
<td>FLS over Wavelet</td>
<td>25.67%</td>
</tr>
<tr>
<td>OLS over Wavelet</td>
<td>15.75%</td>
</tr>
</tbody>
</table>

Table 4: Relative improvement of FLS over OLS and wavelet method.

In order to analyze the impact of the estimation period length on the beta estimates for each interval return, we compute different betas by changing the sample size. We begin with sub-samples of the returns time series which contents the last half of observations and we increase the size of the sample by adding one by one other observation.

The Figures depicting the beta ($\beta(n) = \beta_n$) estimates as a function of the length on the estimation period, shows that the beta estimates depend on the length of training sample for every considered interval return as well for the OLS as for FLS method. These plots indicate that for aggressive stocks, i.e. with systematic risk higher than 1, the beta estimated generally seems to decrease toward 1. For defensive stocks, with beta lower than 1, it is tenuous to read a precise sense of variation even if we observe a kind of increasing trend toward 1.

For testing this hypothesis on the convergence of the beta estimates with sample size as argument, we introduce the following linear model

$$|1 - \beta_n| = \gamma_0 + \gamma_1 n + \epsilon_n, \ n > n_0$$

where $\beta_n$ is the beta estimate based on $n$-length time series, $\gamma_0$ and $\gamma_1$ real parameters, $n_0$ the minimal length of sample used and $\epsilon_n$ a white noise for all faisible $n$.

The model states that there is a linear relationship between the absolute deviation from 1 of the beta estimate and the length estimation sample. Testing its validity is equivalent to test the hypothesis of the nullity of the parameter $\gamma_0$ using a Fisher-Snedecor F-test. The hypotheses

---

5The coefficient of variation (CV) is a normalized measure of dispersion. It is defined by $CV = \frac{\sigma}{\mu}$, where $\sigma$ is the standard deviation and $\mu$ the mean value.
Figure 1: This plot shows the closeness of the FLS and OLS beta estimates for different return intervals. The wavelet estimates are relatively farther.

Figure 2: This plot highlights the impact of the return interval on the beta estimates. We observe that different values of the beta can be estimated by changing the return interval with the same sample.
Figure 3: Plot of coefficients of variation of beta estimates for different return interval showing that the fuzzy least square method produces slightly more stable estimates than OLS and wavelet multiscaling approaches.

are formulated as follow

\[
\begin{cases}
H_0 : \gamma_1 = 0 \\
versus \\
H_1 : \gamma_1 \neq 0
\end{cases}
\]

(12)

The Tables 6, 7, 8 and 9 report the results of the T-test of the linear dependence between the absolute deviation of the beta from 1 and the size of the estimation sample for different return intervals. Except the triweekly return interval, the hypothesis of the linear dependence between the absolute deviations of the standard systematic risk from 1 and the size of the estimation sample is accepted for at least 87% of the examined stocks with weekly, biweekly and monthly returns. Then, the dependence of the beta estimate and the size of the estimation sample already highlighted is the literature, is corroborated with our sample. Moreover, according to the negative sign of the \(\gamma_1\) estimates in the most cases, the beta estimates seem to converge weakly toward 1 when the sample size grows. For the FLS beta estimates, this behavior is observed at a relatively lower proportion (at most for 80%) for the large return intervals (triweekly and monthly returns). However, as depicted in Figure 5, the FLS beta estimates have generally a lower coefficient of variation when the size sample change.

<table>
<thead>
<tr>
<th>Return interval</th>
<th>OLS Rejection rate</th>
<th>OLS Acceptance rate</th>
<th>FLS Rejection rate</th>
<th>FLS Acceptance rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 week</td>
<td>87%</td>
<td>13%</td>
<td>80%</td>
<td>20%</td>
</tr>
<tr>
<td>2 weeks</td>
<td>87%</td>
<td>13%</td>
<td>53%</td>
<td>47%</td>
</tr>
<tr>
<td>3 weeks</td>
<td>37%</td>
<td>63%</td>
<td>30%</td>
<td>70%</td>
</tr>
<tr>
<td>4 weeks</td>
<td>90%</td>
<td>10%</td>
<td>20%</td>
<td>80%</td>
</tr>
</tbody>
</table>

Table 5: Rejection rate of the null hypothesis \(H_0\) with different return intervals and estimation methods.
Figure 4: Plots of beta estimates ($\beta(n) = \beta_n$) as function of size sample $n$ with weekly (top), biweekly (second), triweekly (third) and monthly (bottom) return intervals.
Figure 5: Plot of coefficients of variation of beta estimates for different size samples showing that the fuzzy least square method produces slightly more stable estimates.

6 Conclusions
This paper has focused on the effect of return intervals and on the impact of period length on beta estimates. In order to do that, the return over an interval period is first represented as a fuzzy random variable by associating the return based on closing prices to intraperiod volatility. Next, the one-factor market model is introduced as a fuzzy linear regression model and validated by a hypothesis test based on a bootstrap approach and on an asymptotic approach. The beta from this FML is then estimated by the fuzzy least square method. A theoretical relationship shows how these new beta estimates are a generalized form of the standard systematic risk in some particular cases.

Secondly, a comparative empirical analysis based on 30 French stocks allows showing that the values of FLS beta estimate and OLS beta estimates are generally very close. They are both sensitive to the change of the return intervals and of estimation period length. However, the coefficient of variation of the FLS beta estimates is on average lower than the OLS beta estimates and the wavelet beta estimates when the returns intervals change. Moreover, the linear dependence hypothesis between the absolute deviations of the standard systematic risk from 1 and the size of the estimation sample is accepted for at least 87% of the considered stocks with weekly, biweekly and monthly returns whereas the acceptance rate is very low (between 10% and 13% for weekly, biweekly and monthly return) for the FML-based systematic risk. It also appears that the betas obtained via wavelet methods of Gencay et al. (2005) are generally quite different to the FLS and OLS based ones (which are close). To go into detail of this appearance, in future work it will be interesting to adapt the wavelet multiscaling method to fuzzy returns for computing the systematic risk. A comparative analysis with previous methods could give an explanation of observed differences between wavelet beta estimates and the two others. In
conclusion, this paper has introduced a new estimation approach of the systematic risk based on the fuzzy set theory which deals better with the return interval effect and the impact of the training sample size.

A  Fuzzy set theory: a brief overview

Before proceeding to formal presentation of fuzzy adjusted performance measures, we first briefly review three of the basic concepts of fuzzy theory; namely fuzzy sets, fuzzy numbers and fuzzy random variables. Readers familiar with these topics can skip this section, and those interested in a detailed presentation of fuzzy theory, may see Zimmermann(2001).

A.1 Fuzzy sets and fuzzy numbers

Let \( X \) a crisp set whose elements are denoted \( x \). A fuzzy subset \( A \) of \( X \) is defined by its membership function \( \mu_A : X \to [0, 1] \) which associates each element \( x \) of \( X \) with its membership degree \( \mu_A(x) \) (Zadeh (1965)). The membership function is an extension of the characteristic function of a classical set. It has the possibility to take intermediate values from 0 to 1, in which case some elements can be said to 'belong more' to the set than some others. The degree of membership of an element \( x \) to a fuzzy set \( A \) is equal to 0 (respectively 1) if we want to express with certainty that \( x \) does not belong (respectively belongs) to \( A \).

The crisp set of elements that belong to the fuzzy set \( A \) at least to the degree \( \alpha \) is called the \( \alpha \)-cut or \( \alpha \)-level set and defined by:

\[
A_{\alpha} = \{ x \in X | \mu_A(x) \geq \alpha \}
\]  

\( A_0 \) is the closure\(^8\) of the support\(^9\) of \( A \).

Fuzzy numbers are numbers that have fuzzy properties, examples of which are the notions of "around ten percent" and "extremely low". Dubois and Prade (1980, p. 26) characterizes the fuzzy numbers as follows

**Definition A.1** A fuzzy subset \( A \) of \( \mathbb{R} \) with membership \( \mu_A : \mathbb{R} \to [0, 1] \) is called fuzzy number if

1. \( A \) is normal, i.e. \( \exists \ x_0 \in \mathbb{R} | \mu_A(x_0) = 1 \);

2. \( A \) is fuzzy convex, i.e.

\[
\forall \ x_1, x_2 \in \mathbb{R} | \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \ \forall \lambda \in [0, 1];
\]

\(^8\)The closure of the support of \( A \) is the smallest closed interval containing the support of \( A \) (A definition adapted from Shapiro (2009))

\(^9\)The support of \( A \) is the set of all \( x \) such that \( \mu_A(x) > 0 \). (A definition adapted from Shapiro (2009))
3. $\mu_A$ is upper semi continuous\footnote{The semi-continuity is a weak form of continuity. Intuitively, a function f is called upper semi-continuous at point $x_0$ if the function values for arguments near $x_0$ are either close to $f(x_0)$ or less than $f(x_0)$}.

4. $\text{supp}(A)$ is bounded.

**Example A.2** (Zimmermann(1996, p. 64)) A LR-fuzzy number, denoted by $\tilde{A} = \langle l, c, r \rangle_{LR}$, where $c \in \mathbb{R}^+$ is called central value, and $l \in \mathbb{R}^+$ and $r \in \mathbb{R}^+$ is the left and the right spread, respectively, is characterized by a membership function of the form

$$\mu_A(x) = \begin{cases} L\left(\frac{x-c}{l}\right) & \text{if } c - l \leq x \leq c, \\ R\left(\frac{x-c}{r}\right) & \text{if } r + c \geq x \geq c, \\ 0 & \text{else} \end{cases}$$

(14)

$L : \mathbb{R}^+ \rightarrow [0, 1], R : \mathbb{R}^+ \rightarrow [0, 1]$ are strictly continuous decreasing functions such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. $L$ and $R$ are called the left and the right shape functions respectively. If right and left spreads are equal and $L := R$, the LR-fuzzy number is said to be a symmetric fuzzy number and denoted $\tilde{A} = (c, \Delta)$. $\Delta$ is the spread equal to $l = r$.

We limit the present study to triangular fuzzy numbers characterized by the shape function $R(x) := L(x) := \max\{1 - x, 0\}$. Extensions to other membership functions need to be addressed in future work.

Using Zadeh’s extension principle (Zadeh (1965)), which is a rule providing a general method to extend a function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ on the set of fuzzy numbers, we can define binary operator such as addition, subtraction, multiplication... for two fuzzy set. When $k = 2$, this method allows defining the membership function of the result as follows

$$\mu_{\tilde{A}_1 \circ \tilde{A}_2}(z) = \sup_{(x_1, x_2) \in \tilde{A}_1 \times \tilde{A}_2} \{\min \left(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)\right) / x_1 \circ x_2 = z\}$$

(15)

where $\circ$ is the binary operator.

Several distances between fuzzy numbers are defined in the literature for very specific purposes. The metric $\delta_2$ introduced by Bertoluzza et al. (1995) in the case of a one-dimensional convex set, is one of the most commonly used for least square problems (Näther (2006), González-Rodríguez et al. (2009)). For two symmetric LR-fuzzy numbers $\tilde{A}_i = (a_i, \Delta_i), i = 1, 2$ , with shape function $L : \mathbb{R}^+ \rightarrow [0, 1]$, it is defined by:

$$\delta_2(A_1, A_2) = \sqrt{(a_1 - a_2)^2 + l(\Delta_1 - \Delta_2)^2}$$

(16)

where

$$l = \int_0^1 (L^{-1}(\alpha))^2 d(\alpha)$$

In the case of a symmetric LR-fuzzy number, $l = 1/3$.\footnote{The semi-continuity is a weak form of continuity. Intuitively, a function f is called upper semi-continuous at point $x_0$ if the function values for arguments near $x_0$ are either close to $f(x_0)$ or less than $f(x_0)$}
A.2 Fuzzy random variables

Different approaches of the concept of fuzzy random variables have been developed in the literature since 70’s. The most often cited being introduced by Kwakernaak (1978) and enhanced by Kruse and Meyer (1987), and the one by Puri and Ralescu (1986). An extensive discussion on these two approaches is given by Shapiro (2009). For the purpose of this study, we adopt the concept of FRVs as ‘fuzzy perception/observation/report of a classical real-valued random variable’ of Kwarkernaak/Kruze and Mayer.

Let $\mathcal{F}(\mathbb{R})_c$ denotes the set of all normal convex fuzzy subsets of $\mathbb{R}$ and $(\Omega, \mathcal{A}, P)$ a probability space.

More precisely, Kruse and Meyer (1987) have defined a FRV as follows

Definition A.3 The mapping $X : \Omega \rightarrow \mathcal{F}(\mathbb{R})_c$ is said to be a FRV if for all $\alpha \in [0, 1]$, the two real-valued mappings $X^L_\alpha \rightarrow \mathbb{R}$ and $X^R_\alpha \rightarrow \mathbb{R}$ (defined so that for all $\omega \in \Omega$ we have that $[X^L_\alpha(\omega), X^R_\alpha(\omega)]$) are real-valued random variables.

Puri and Ralescu (1986) brought an expectation operator called the Aumann-Expectation and denoted $E^A$ for FRVs. Its construction is based on the Aumann’s (1965) study on integrals of interval-valued functions. For a symmetric LR-fuzzy random variable $\tilde{A} = \langle a, \Delta \rangle$, the Aumann-expectation $E^A$ is defined by (Körner (1997)):

$$E^A[\tilde{A}] = (E[a], E[\Delta])$$

(17)

The definition shows that the Aumann-expectation is a linear operator as the expectation operator for real random variables.

Following the Kruse and Meyer’s definition of a FRV and based on the definition Körner (1997) introduced a real-valued variance characterized by the Fréchet principle and covariance.

For LR-fuzzy numbers $\tilde{A}_i = \langle a_i, \Delta_i \rangle$, $i = 1, 2$, Körner (1997) describes the variance and covariance operators as follow:

$$Cov^A[\tilde{A}_1, \tilde{A}_2] = Cov[a_1, a_2] + lCov[\Delta_1, \Delta_2],$$

$$Var^A[\tilde{A}] = Var[a] + lVar[\Delta].$$

---

11 A fuzzy set $\tilde{A}$ is called a normal convex fuzzy subset of $\mathbb{R}$ if $\tilde{A}$ is normal, the $\alpha$-cuts of $\tilde{A}$ are convex and compact and the support of $\tilde{A}$ is compact. (Körner(1997))

12 Where $\Omega$ is the set of all possible outcomes described by the probability space, $\mathcal{A}$ is $\sigma$-fields of subsets of $\Omega$, and the function $P$ defined on $\mathcal{A}$ is a probability measure.

13 Fréchet defined in 1948 the expectation operator $E^{(d)}Z$ and the variance $Var^{(d)}Z$ for a random variable $Z$ in a metric space $(M, d)$ as

$$E[d^2(Z, E^{(d)}Z)] = \inf_{x \in M} E[d^2(Z, x)],$$

$$Var^{(d)}[Z] = E[d^2(Z, E^{(d)}Z)].$$
where \( l = \int_0^1 (L^{-1}(\alpha))^2 d(\alpha) \).

in the case of symmetric fuzzy triangular numbers

### B Tables

<table>
<thead>
<tr>
<th>Stocks</th>
<th>( \hat{\gamma}_1 )</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
<th>( \hat{\gamma}_1 )</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acc</td>
<td>-0.0062</td>
<td>0.0010</td>
<td>-6.2994</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0005</td>
<td>0.0041</td>
<td>0.1311</td>
<td>0.9040</td>
<td>Accp</td>
</tr>
<tr>
<td>AirL</td>
<td>0.0038</td>
<td>0.0009</td>
<td>4.1374</td>
<td>0.0009</td>
<td>Rej</td>
<td>0.0147</td>
<td>0.0063</td>
<td>2.3392</td>
<td>0.1013</td>
<td>Accp</td>
</tr>
<tr>
<td>ALC-L</td>
<td>-0.0114</td>
<td>0.0016</td>
<td>-7.0772</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0087</td>
<td>0.0034</td>
<td>2.5600</td>
<td>0.0832</td>
<td>Accp</td>
</tr>
<tr>
<td>Alli</td>
<td>-0.0055</td>
<td>0.0012</td>
<td>-4.7071</td>
<td>0.0033</td>
<td>Rej</td>
<td>-0.0195</td>
<td>0.0114</td>
<td>-1.7146</td>
<td>0.1846</td>
<td>Accp</td>
</tr>
<tr>
<td>ArcMit</td>
<td>-0.0236</td>
<td>0.0037</td>
<td>-6.3794</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0073</td>
<td>0.0084</td>
<td>-0.8745</td>
<td>0.4462</td>
<td>Accp</td>
</tr>
<tr>
<td>AXA</td>
<td>-0.0093</td>
<td>0.0008</td>
<td>-11.6099</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0157</td>
<td>0.0074</td>
<td>-2.1309</td>
<td>0.1229</td>
<td>Accp</td>
</tr>
<tr>
<td>BNP</td>
<td>-0.0009</td>
<td>0.0008</td>
<td>-1.1462</td>
<td>0.2697</td>
<td>Accp</td>
<td>-0.0026</td>
<td>0.0069</td>
<td>-0.3797</td>
<td>0.7295</td>
<td>Accp</td>
</tr>
<tr>
<td>BOUY</td>
<td>0.0029</td>
<td>0.0006</td>
<td>4.9548</td>
<td>0.0002</td>
<td>Rej</td>
<td>-0.0001</td>
<td>0.0032</td>
<td>-0.0426</td>
<td>0.9687</td>
<td>Accp</td>
</tr>
<tr>
<td>CapG</td>
<td>-0.0101</td>
<td>0.0015</td>
<td>-6.8041</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0040</td>
<td>0.0011</td>
<td>3.6765</td>
<td>0.0348</td>
<td>Accp</td>
</tr>
<tr>
<td>Carr</td>
<td>-0.0005</td>
<td>0.0015</td>
<td>-0.3047</td>
<td>0.7648</td>
<td>Accp</td>
<td>-0.0211</td>
<td>0.0144</td>
<td>-1.4581</td>
<td>0.2409</td>
<td>Accp</td>
</tr>
<tr>
<td>CA</td>
<td>0.0003</td>
<td>0.0010</td>
<td>0.3594</td>
<td>0.7243</td>
<td>Accp</td>
<td>-0.0043</td>
<td>0.0086</td>
<td>-0.5029</td>
<td>0.6496</td>
<td>Accp</td>
</tr>
<tr>
<td>Dan</td>
<td>-0.0045</td>
<td>0.0017</td>
<td>-2.6292</td>
<td>0.0190</td>
<td>Rej</td>
<td>-0.0276</td>
<td>0.0123</td>
<td>-2.2343</td>
<td>0.1115</td>
<td>Accp</td>
</tr>
<tr>
<td>Dex</td>
<td>-0.0134</td>
<td>0.0022</td>
<td>-6.1977</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0118</td>
<td>0.0069</td>
<td>1.7087</td>
<td>0.1860</td>
<td>Accp</td>
</tr>
<tr>
<td>EDF</td>
<td>-0.0050</td>
<td>0.0011</td>
<td>-4.6407</td>
<td>0.0033</td>
<td>Rej</td>
<td>-0.0113</td>
<td>0.0061</td>
<td>-1.8508</td>
<td>0.1613</td>
<td>Accp</td>
</tr>
<tr>
<td>GDFS</td>
<td>-0.0043</td>
<td>0.0004</td>
<td>-11.2106</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0054</td>
<td>0.0013</td>
<td>-4.0034</td>
<td>0.0279</td>
<td>Accp</td>
</tr>
<tr>
<td>Laf</td>
<td>-0.0128</td>
<td>0.0019</td>
<td>-6.6825</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0023</td>
<td>0.0024</td>
<td>-0.9556</td>
<td>0.4098</td>
<td>Accp</td>
</tr>
<tr>
<td>Lag</td>
<td>-0.0037</td>
<td>0.0013</td>
<td>-2.7980</td>
<td>0.0135</td>
<td>Rej</td>
<td>0.0053</td>
<td>0.0100</td>
<td>0.5315</td>
<td>0.6319</td>
<td>Accp</td>
</tr>
<tr>
<td>LOre</td>
<td>-0.0025</td>
<td>0.0005</td>
<td>-5.2675</td>
<td>0.0001</td>
<td>Rej</td>
<td>0.0026</td>
<td>0.0043</td>
<td>0.5949</td>
<td>0.5938</td>
<td>Accp</td>
</tr>
<tr>
<td>LVMH</td>
<td>-0.0040</td>
<td>0.0007</td>
<td>-5.9157</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0017</td>
<td>0.0016</td>
<td>-1.0569</td>
<td>0.3681</td>
<td>Accp</td>
</tr>
<tr>
<td>Mich</td>
<td>-0.0092</td>
<td>0.0015</td>
<td>-6.2130</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0171</td>
<td>0.0037</td>
<td>-4.6819</td>
<td>0.0184</td>
<td>Accp</td>
</tr>
<tr>
<td>Pen</td>
<td>-0.0185</td>
<td>0.0021</td>
<td>-9.0225</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0139</td>
<td>0.0041</td>
<td>-3.3842</td>
<td>0.0430</td>
<td>Accp</td>
</tr>
<tr>
<td>Ren</td>
<td>-0.0142</td>
<td>0.0017</td>
<td>-8.3166</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0243</td>
<td>0.0056</td>
<td>-4.3396</td>
<td>0.0226</td>
<td>Accp</td>
</tr>
<tr>
<td>SG</td>
<td>-0.0040</td>
<td>0.0011</td>
<td>-3.7536</td>
<td>0.0019</td>
<td>Rej</td>
<td>-0.0167</td>
<td>0.0076</td>
<td>-2.1940</td>
<td>0.1158</td>
<td>Accp</td>
</tr>
<tr>
<td>STM</td>
<td>-0.0046</td>
<td>0.0014</td>
<td>-3.3814</td>
<td>0.0041</td>
<td>Rej</td>
<td>0.0118</td>
<td>0.0044</td>
<td>2.6476</td>
<td>0.0771</td>
<td>Accp</td>
</tr>
<tr>
<td>Tech</td>
<td>-0.0246</td>
<td>0.0033</td>
<td>-7.3946</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0089</td>
<td>0.0031</td>
<td>-2.8692</td>
<td>0.0641</td>
<td>Accp</td>
</tr>
<tr>
<td>Tot</td>
<td>-0.0061</td>
<td>0.0007</td>
<td>-8.4122</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0021</td>
<td>0.0030</td>
<td>-0.6944</td>
<td>0.5374</td>
<td>Accp</td>
</tr>
<tr>
<td>Vail</td>
<td>-0.0179</td>
<td>0.0026</td>
<td>-6.9527</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0242</td>
<td>0.0130</td>
<td>-1.8576</td>
<td>0.1602</td>
<td>Accp</td>
</tr>
<tr>
<td>VEOE</td>
<td>-0.0107</td>
<td>0.0020</td>
<td>-5.3264</td>
<td>0.0001</td>
<td>Rej</td>
<td>-0.0089</td>
<td>0.0106</td>
<td>-0.8422</td>
<td>0.4615</td>
<td>Accp</td>
</tr>
<tr>
<td>Vin</td>
<td>-0.0048</td>
<td>0.0008</td>
<td>-5.6632</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0070</td>
<td>0.0048</td>
<td>1.4472</td>
<td>0.2437</td>
<td>Accp</td>
</tr>
<tr>
<td>Viv</td>
<td>-0.0066</td>
<td>0.0011</td>
<td>-6.2530</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0052</td>
<td>0.0014</td>
<td>-3.7060</td>
<td>0.0341</td>
<td>Rej</td>
</tr>
</tbody>
</table>

Table 6: Results of the testing the nullity hypothesis of \( \gamma_1 \) based on monthly returns with beta estimated by OLS and FLS.
<table>
<thead>
<tr>
<th>Stocks</th>
<th>$\gamma_1$</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
<th>$\gamma_1$</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acc</td>
<td>0.0012</td>
<td>0.0014</td>
<td>0.8186</td>
<td>0.4502</td>
<td>Rej</td>
<td>0.0009</td>
<td>0.0013</td>
<td>0.6707</td>
<td>0.5321</td>
<td>Rej</td>
</tr>
<tr>
<td>AirL</td>
<td>0.0044</td>
<td>0.0018</td>
<td>2.4512</td>
<td>0.0579</td>
<td>Rej</td>
<td>0.0037</td>
<td>0.0016</td>
<td>2.3275</td>
<td>0.0674</td>
<td>Rej</td>
</tr>
<tr>
<td>Alc-L</td>
<td>0.0049</td>
<td>0.0013</td>
<td>3.7278</td>
<td>0.0136</td>
<td>Rej</td>
<td>0.0042</td>
<td>0.0011</td>
<td>3.7728</td>
<td>0.0130</td>
<td>Rej</td>
</tr>
<tr>
<td>Alli</td>
<td>0.0014</td>
<td>0.0043</td>
<td>0.3235</td>
<td>0.7594</td>
<td>Rej</td>
<td>0.0018</td>
<td>0.0039</td>
<td>0.4618</td>
<td>0.6636</td>
<td>Rej</td>
</tr>
<tr>
<td>ArcMit</td>
<td>-0.0075</td>
<td>0.0044</td>
<td>-1.7095</td>
<td>0.1481</td>
<td>Rej</td>
<td>-0.0054</td>
<td>0.0041</td>
<td>-1.3253</td>
<td>0.2424</td>
<td>Rej</td>
</tr>
<tr>
<td>AXA</td>
<td>0.0033</td>
<td>0.0032</td>
<td>1.0408</td>
<td>0.3457</td>
<td>Rej</td>
<td>0.0042</td>
<td>0.0028</td>
<td>1.5091</td>
<td>0.1916</td>
<td>Rej</td>
</tr>
<tr>
<td>BNP</td>
<td>0.0061</td>
<td>0.0032</td>
<td>1.8959</td>
<td>0.1165</td>
<td>Rej</td>
<td>0.0067</td>
<td>0.0032</td>
<td>2.0814</td>
<td>0.0919</td>
<td>Rej</td>
</tr>
<tr>
<td>BOUY</td>
<td>-0.0002</td>
<td>0.0020</td>
<td>-0.1140</td>
<td>0.9137</td>
<td>Rej</td>
<td>-0.0007</td>
<td>0.0018</td>
<td>-0.4029</td>
<td>0.7036</td>
<td>Rej</td>
</tr>
<tr>
<td>CapG</td>
<td>0.0076</td>
<td>0.0014</td>
<td>5.3260</td>
<td>0.0031</td>
<td>Rej</td>
<td>0.0063</td>
<td>0.0015</td>
<td>4.3313</td>
<td>0.0075</td>
<td>Rej</td>
</tr>
<tr>
<td>Carr</td>
<td>0.0008</td>
<td>0.0031</td>
<td>0.2722</td>
<td>0.7963</td>
<td>Rej</td>
<td>0.0005</td>
<td>0.0028</td>
<td>0.1759</td>
<td>0.8673</td>
<td>Rej</td>
</tr>
<tr>
<td>CA</td>
<td>0.0016</td>
<td>0.0037</td>
<td>0.4293</td>
<td>0.6856</td>
<td>Rej</td>
<td>0.0015</td>
<td>0.0035</td>
<td>0.4236</td>
<td>0.6895</td>
<td>Rej</td>
</tr>
<tr>
<td>Dan</td>
<td>-0.0075</td>
<td>0.0034</td>
<td>-2.2142</td>
<td>0.0777</td>
<td>Rej</td>
<td>-0.0074</td>
<td>0.0030</td>
<td>-2.4442</td>
<td>0.0584</td>
<td>Rej</td>
</tr>
<tr>
<td>Dex</td>
<td>0.0065</td>
<td>0.0033</td>
<td>1.9769</td>
<td>0.1050</td>
<td>Rej</td>
<td>0.0088</td>
<td>0.0029</td>
<td>3.0560</td>
<td>0.0282</td>
<td>Rej</td>
</tr>
<tr>
<td>EDF</td>
<td>0.0007</td>
<td>0.0029</td>
<td>0.2381</td>
<td>0.8213</td>
<td>Rej</td>
<td>0.0004</td>
<td>0.0026</td>
<td>0.1431</td>
<td>0.8918</td>
<td>Rej</td>
</tr>
<tr>
<td>GDFS</td>
<td>0.0054</td>
<td>0.0010</td>
<td>5.5156</td>
<td>0.0027</td>
<td>Rej</td>
<td>0.0055</td>
<td>0.0010</td>
<td>5.6125</td>
<td>0.0025</td>
<td>Rej</td>
</tr>
<tr>
<td>Laf</td>
<td>-0.0114</td>
<td>0.0020</td>
<td>-5.7136</td>
<td>0.0023</td>
<td>Rej</td>
<td>-0.0102</td>
<td>0.0017</td>
<td>-6.0601</td>
<td>0.0018</td>
<td>Rej</td>
</tr>
<tr>
<td>Lag</td>
<td>-0.0032</td>
<td>0.0043</td>
<td>-0.7403</td>
<td>0.4924</td>
<td>Rej</td>
<td>-0.0026</td>
<td>0.0040</td>
<td>-0.6563</td>
<td>0.5406</td>
<td>Rej</td>
</tr>
<tr>
<td>LOre</td>
<td>-0.0038</td>
<td>0.0018</td>
<td>-2.0723</td>
<td>0.0930</td>
<td>Rej</td>
<td>-0.0032</td>
<td>0.0018</td>
<td>-1.7523</td>
<td>0.1401</td>
<td>Rej</td>
</tr>
<tr>
<td>LVMH</td>
<td>0.0033</td>
<td>0.0007</td>
<td>4.6972</td>
<td>0.0054</td>
<td>Rej</td>
<td>0.0033</td>
<td>0.0007</td>
<td>4.9010</td>
<td>0.0045</td>
<td>Rej</td>
</tr>
<tr>
<td>Mich</td>
<td>0.0004</td>
<td>0.0027</td>
<td>0.1490</td>
<td>0.8874</td>
<td>Rej</td>
<td>-0.0004</td>
<td>0.0024</td>
<td>-0.1602</td>
<td>0.8790</td>
<td>Rej</td>
</tr>
<tr>
<td>Pen</td>
<td>-0.0060</td>
<td>0.0023</td>
<td>-2.6126</td>
<td>0.0475</td>
<td>Rej</td>
<td>-0.0049</td>
<td>0.0022</td>
<td>-2.2632</td>
<td>0.0731</td>
<td>Rej</td>
</tr>
<tr>
<td>Ren</td>
<td>-0.0053</td>
<td>0.0008</td>
<td>-6.5686</td>
<td>0.0012</td>
<td>Rej</td>
<td>-0.0049</td>
<td>0.0008</td>
<td>-6.2436</td>
<td>0.0015</td>
<td>Rej</td>
</tr>
<tr>
<td>SG</td>
<td>-0.0050</td>
<td>0.0014</td>
<td>-3.6169</td>
<td>0.0153</td>
<td>Rej</td>
<td>-0.0039</td>
<td>0.0013</td>
<td>-2.9292</td>
<td>0.0327</td>
<td>Rej</td>
</tr>
<tr>
<td>STM</td>
<td>0.0129</td>
<td>0.0021</td>
<td>6.2013</td>
<td>0.0016</td>
<td>Rej</td>
<td>0.0307</td>
<td>0.0029</td>
<td>1.3081</td>
<td>0.2478</td>
<td>Rej</td>
</tr>
<tr>
<td>Tech</td>
<td>-0.0055</td>
<td>0.0031</td>
<td>-1.7641</td>
<td>0.1380</td>
<td>Rej</td>
<td>-0.0018</td>
<td>0.0026</td>
<td>-0.7017</td>
<td>0.5141</td>
<td>Rej</td>
</tr>
<tr>
<td>Tot</td>
<td>0.0021</td>
<td>0.0018</td>
<td>1.2000</td>
<td>0.2839</td>
<td>Rej</td>
<td>0.0019</td>
<td>0.0017</td>
<td>1.1215</td>
<td>0.3130</td>
<td>Rej</td>
</tr>
<tr>
<td>Vall</td>
<td>-0.0099</td>
<td>0.0043</td>
<td>-2.2952</td>
<td>0.0702</td>
<td>Rej</td>
<td>-0.0087</td>
<td>0.0040</td>
<td>-2.1545</td>
<td>0.0838</td>
<td>Rej</td>
</tr>
<tr>
<td>VeoE</td>
<td>-0.0014</td>
<td>0.0050</td>
<td>-0.2844</td>
<td>0.7875</td>
<td>Rej</td>
<td>-0.0021</td>
<td>0.0046</td>
<td>-0.4553</td>
<td>0.6680</td>
<td>Rej</td>
</tr>
<tr>
<td>Vin</td>
<td>0.0055</td>
<td>0.0015</td>
<td>3.6832</td>
<td>0.0142</td>
<td>Rej</td>
<td>0.0048</td>
<td>0.0013</td>
<td>3.5639</td>
<td>0.0161</td>
<td>Rej</td>
</tr>
<tr>
<td>Viv</td>
<td>-0.0030</td>
<td>0.0011</td>
<td>-2.6746</td>
<td>0.0441</td>
<td>Rej</td>
<td>-0.0022</td>
<td>0.0011</td>
<td>-1.9573</td>
<td>0.1077</td>
<td>Rej</td>
</tr>
</tbody>
</table>

Table 7: Results of the testing the nullity hypothesis of $\gamma_1$ based on triweekly returns with beta estimated by OLS and FLS.
<table>
<thead>
<tr>
<th>Stocks</th>
<th>$\gamma_1$</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
<th>$\gamma_1$</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acc</td>
<td>0.0004</td>
<td>0.0002</td>
<td>1.8144</td>
<td>0.0790</td>
<td>Rej</td>
<td>0.0006</td>
<td>0.0010</td>
<td>0.5841</td>
<td>0.5735</td>
<td>Rej</td>
</tr>
<tr>
<td>AirL</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.2782</td>
<td>0.7826</td>
<td>Rej</td>
<td>0.0026</td>
<td>0.0003</td>
<td>7.7728</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Alc-L</td>
<td>-0.0041</td>
<td>0.0003</td>
<td>-12.6420</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0021</td>
<td>0.0005</td>
<td>3.8124</td>
<td>0.0041</td>
<td>Rej</td>
</tr>
<tr>
<td>Alli</td>
<td>-0.0008</td>
<td>0.0001</td>
<td>-5.5944</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0035</td>
<td>0.0006</td>
<td>-5.4977</td>
<td>0.0004</td>
<td>Rej</td>
</tr>
<tr>
<td>ArcMit</td>
<td>-0.0029</td>
<td>0.0004</td>
<td>-7.6138</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0025</td>
<td>0.0010</td>
<td>2.5260</td>
<td>0.0324</td>
<td>Rej</td>
</tr>
<tr>
<td>AXA</td>
<td>-0.0012</td>
<td>0.0001</td>
<td>-10.2215</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0015</td>
<td>0.0006</td>
<td>-2.7568</td>
<td>0.0222</td>
<td>Rej</td>
</tr>
<tr>
<td>BouY</td>
<td>0.0014</td>
<td>0.0002</td>
<td>6.7526</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0006</td>
<td>0.0009</td>
<td>0.7340</td>
<td>0.4816</td>
<td>Rej</td>
</tr>
<tr>
<td>CapG</td>
<td>-0.0014</td>
<td>0.0002</td>
<td>-6.4183</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0012</td>
<td>0.0004</td>
<td>3.4475</td>
<td>0.0073</td>
<td>Rej</td>
</tr>
<tr>
<td>Carr</td>
<td>0.0008</td>
<td>0.0002</td>
<td>4.3035</td>
<td>0.0001</td>
<td>Rej</td>
<td>-0.0003</td>
<td>0.0013</td>
<td>-0.2028</td>
<td>0.8438</td>
<td>Rej</td>
</tr>
<tr>
<td>CA</td>
<td>-0.0016</td>
<td>0.0001</td>
<td>-14.5924</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0032</td>
<td>0.0006</td>
<td>-5.6831</td>
<td>0.0003</td>
<td>Rej</td>
</tr>
<tr>
<td>Dan</td>
<td>-0.0012</td>
<td>0.0001</td>
<td>-9.3351</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0027</td>
<td>0.0008</td>
<td>-3.2300</td>
<td>0.0103</td>
<td>Rej</td>
</tr>
<tr>
<td>Dex</td>
<td>-0.0051</td>
<td>0.0003</td>
<td>-17.3663</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0010</td>
<td>0.0011</td>
<td>-0.9257</td>
<td>0.3788</td>
<td>Rej</td>
</tr>
<tr>
<td>EDF</td>
<td>-0.0016</td>
<td>0.0002</td>
<td>-8.5516</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0012</td>
<td>0.0009</td>
<td>1.3342</td>
<td>0.2149</td>
<td>Rej</td>
</tr>
<tr>
<td>GDFS</td>
<td>-0.0019</td>
<td>0.0003</td>
<td>-6.3853</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0039</td>
<td>0.0010</td>
<td>-3.7890</td>
<td>0.0043</td>
<td>Rej</td>
</tr>
<tr>
<td>Laf</td>
<td>-0.0031</td>
<td>0.0003</td>
<td>-9.7291</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0020</td>
<td>0.0009</td>
<td>-2.1275</td>
<td>0.0623</td>
<td>Rej</td>
</tr>
<tr>
<td>Lag</td>
<td>-0.0024</td>
<td>0.0002</td>
<td>-9.8451</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0017</td>
<td>0.0014</td>
<td>1.2899</td>
<td>0.2293</td>
<td>Rej</td>
</tr>
<tr>
<td>LOre</td>
<td>-0.0008</td>
<td>0.0002</td>
<td>-3.9916</td>
<td>0.0004</td>
<td>Rej</td>
<td>0.0006</td>
<td>0.0005</td>
<td>1.1818</td>
<td>0.2676</td>
<td>Rej</td>
</tr>
<tr>
<td>LVMH</td>
<td>-0.0006</td>
<td>0.0001</td>
<td>-6.1101</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0005</td>
<td>0.0003</td>
<td>-2.0981</td>
<td>0.0653</td>
<td>Rej</td>
</tr>
<tr>
<td>Mich</td>
<td>-0.0023</td>
<td>0.0002</td>
<td>-10.9008</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0049</td>
<td>0.0006</td>
<td>-8.0268</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Peu</td>
<td>-0.0032</td>
<td>0.0004</td>
<td>-8.8945</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0036</td>
<td>0.0005</td>
<td>-6.6334</td>
<td>0.0001</td>
<td>Rej</td>
</tr>
<tr>
<td>Ren</td>
<td>-0.0033</td>
<td>0.0003</td>
<td>-9.6065</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0058</td>
<td>0.0008</td>
<td>-7.4851</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>SG</td>
<td>-0.0010</td>
<td>0.0003</td>
<td>-3.4373</td>
<td>0.0016</td>
<td>Rej</td>
<td>-0.0038</td>
<td>0.0008</td>
<td>-4.5229</td>
<td>0.0014</td>
<td>Rej</td>
</tr>
<tr>
<td>STM</td>
<td>-0.0011</td>
<td>0.0002</td>
<td>-5.7626</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0057</td>
<td>0.0009</td>
<td>-6.5827</td>
<td>0.0001</td>
<td>Rej</td>
</tr>
<tr>
<td>Tech</td>
<td>-0.0089</td>
<td>0.0007</td>
<td>-13.4012</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0015</td>
<td>0.0009</td>
<td>1.7085</td>
<td>0.1217</td>
<td>Rej</td>
</tr>
<tr>
<td>Tot</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-2.2808</td>
<td>0.0288</td>
<td>Rej</td>
<td>-0.0010</td>
<td>0.0005</td>
<td>-2.1400</td>
<td>0.0610</td>
<td>Rej</td>
</tr>
<tr>
<td>Vail</td>
<td>-0.0026</td>
<td>0.0004</td>
<td>-6.6688</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0027</td>
<td>0.0019</td>
<td>-1.4207</td>
<td>0.1891</td>
<td>Rej</td>
</tr>
<tr>
<td>VeoE</td>
<td>0.0017</td>
<td>0.0002</td>
<td>7.9424</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0023</td>
<td>0.0013</td>
<td>1.7256</td>
<td>0.1185</td>
<td>Rej</td>
</tr>
<tr>
<td>Vin</td>
<td>0.0001</td>
<td>0.0001</td>
<td>1.5717</td>
<td>0.1258</td>
<td>Rej</td>
<td>0.0016</td>
<td>0.0004</td>
<td>4.5629</td>
<td>0.0014</td>
<td>Rej</td>
</tr>
<tr>
<td>Viv</td>
<td>-0.0027</td>
<td>0.0004</td>
<td>-7.2200</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0025</td>
<td>0.0009</td>
<td>-2.7693</td>
<td>0.0218</td>
<td>Rej</td>
</tr>
</tbody>
</table>

Table 8: Results of the testing the nullity hypothesis of $\gamma_1$ based on biweekly returns with beta estimated by OLS and FLS.
<table>
<thead>
<tr>
<th>Stocks</th>
<th>( \gamma_1 )</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
<th>( \gamma_1 )</th>
<th>Stand. Dev</th>
<th>F-stat</th>
<th>P-value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acc</td>
<td>-0.0009</td>
<td>0.0004</td>
<td>-2.5351</td>
<td>0.0193</td>
<td>Rej</td>
<td>-0.0008</td>
<td>0.0003</td>
<td>-2.3622</td>
<td>0.0279</td>
<td>Rej</td>
</tr>
<tr>
<td>AirL</td>
<td>0.0011</td>
<td>0.0001</td>
<td>18.8014</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0009</td>
<td>0.0000</td>
<td>18.2079</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Alc-L</td>
<td>0.0010</td>
<td>0.0001</td>
<td>11.2837</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0006</td>
<td>0.0001</td>
<td>8.1142</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Ali</td>
<td>-0.0009</td>
<td>0.0002</td>
<td>-3.9150</td>
<td>0.0008</td>
<td>Rej</td>
<td>-0.0006</td>
<td>0.0002</td>
<td>-2.8093</td>
<td>0.0105</td>
<td>Rej</td>
</tr>
<tr>
<td>ArcMit</td>
<td>-0.0035</td>
<td>0.0004</td>
<td>-7.8846</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0028</td>
<td>0.0004</td>
<td>-7.5376</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>AXA</td>
<td>0.0005</td>
<td>0.0002</td>
<td>2.2576</td>
<td>0.0347</td>
<td>Rej</td>
<td>0.0010</td>
<td>0.0002</td>
<td>5.8829</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>BNP</td>
<td>0.0007</td>
<td>0.0003</td>
<td>2.2135</td>
<td>0.0381</td>
<td>Rej</td>
<td>0.0012</td>
<td>0.0003</td>
<td>4.3587</td>
<td>0.0003</td>
<td>Rej</td>
</tr>
<tr>
<td>BOUY</td>
<td>0.0014</td>
<td>0.0002</td>
<td>8.8695</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0010</td>
<td>0.0001</td>
<td>7.1717</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>CapG</td>
<td>0.0026</td>
<td>0.0002</td>
<td>11.8925</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0021</td>
<td>0.0002</td>
<td>10.1537</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Carr</td>
<td>-0.0003</td>
<td>0.0003</td>
<td>-1.3376</td>
<td>0.1953</td>
<td>Rej</td>
<td>-0.0005</td>
<td>0.0002</td>
<td>-2.5440</td>
<td>0.0189</td>
<td>Rej</td>
</tr>
<tr>
<td>CA</td>
<td>-0.0012</td>
<td>0.0003</td>
<td>-3.3387</td>
<td>0.0031</td>
<td>Rej</td>
<td>-0.0009</td>
<td>0.0003</td>
<td>-3.1103</td>
<td>0.0053</td>
<td>Rej</td>
</tr>
<tr>
<td>Dan</td>
<td>0.0001</td>
<td>0.0004</td>
<td>0.3446</td>
<td>0.7338</td>
<td>Rej</td>
<td>-0.0001</td>
<td>0.0003</td>
<td>-0.4143</td>
<td>0.6828</td>
<td>Rej</td>
</tr>
<tr>
<td>Dex</td>
<td>-0.0009</td>
<td>0.0003</td>
<td>-3.1882</td>
<td>0.0044</td>
<td>Rej</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.7593</td>
<td>0.4561</td>
<td>Rej</td>
</tr>
<tr>
<td>EDF</td>
<td>0.0006</td>
<td>0.0002</td>
<td>2.7077</td>
<td>0.0132</td>
<td>Rej</td>
<td>0.0005</td>
<td>0.0002</td>
<td>2.7340</td>
<td>0.0124</td>
<td>Rej</td>
</tr>
<tr>
<td>GDFS</td>
<td>0.0013</td>
<td>0.0002</td>
<td>5.3368</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0004</td>
<td>0.0002</td>
<td>1.5729</td>
<td>0.1307</td>
<td>Rej</td>
</tr>
<tr>
<td>Laf</td>
<td>-0.0013</td>
<td>0.0002</td>
<td>-6.5707</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0010</td>
<td>0.0002</td>
<td>-5.3010</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Lag</td>
<td>0.0008</td>
<td>0.0002</td>
<td>4.2962</td>
<td>0.0003</td>
<td>Rej</td>
<td>0.0008</td>
<td>0.0002</td>
<td>4.9542</td>
<td>0.0001</td>
<td>Rej</td>
</tr>
<tr>
<td>LOre</td>
<td>-0.0004</td>
<td>0.0001</td>
<td>-3.8284</td>
<td>0.0010</td>
<td>Rej</td>
<td>-0.0002</td>
<td>0.0001</td>
<td>-2.5862</td>
<td>0.0172</td>
<td>Rej</td>
</tr>
<tr>
<td>LVMH</td>
<td>0.0013</td>
<td>0.0001</td>
<td>12.9486</td>
<td>0.0000</td>
<td>Rej</td>
<td>0.0013</td>
<td>0.0001</td>
<td>16.8943</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Mich</td>
<td>0.0006</td>
<td>0.0003</td>
<td>1.5867</td>
<td>0.1275</td>
<td>Rej</td>
<td>0.0002</td>
<td>0.0003</td>
<td>0.4908</td>
<td>0.6287</td>
<td>Rej</td>
</tr>
<tr>
<td>Pen</td>
<td>-0.0008</td>
<td>0.0003</td>
<td>-2.5692</td>
<td>0.0179</td>
<td>Rej</td>
<td>-0.0005</td>
<td>0.0003</td>
<td>-1.7482</td>
<td>0.0950</td>
<td>Rej</td>
</tr>
<tr>
<td>Ren</td>
<td>-0.0008</td>
<td>0.0003</td>
<td>-3.0236</td>
<td>0.0065</td>
<td>Rej</td>
<td>-0.0006</td>
<td>0.0002</td>
<td>-2.5443</td>
<td>0.0189</td>
<td>Rej</td>
</tr>
<tr>
<td>SG</td>
<td>-0.0016</td>
<td>0.0003</td>
<td>-4.8760</td>
<td>0.0001</td>
<td>Rej</td>
<td>-0.0010</td>
<td>0.0003</td>
<td>-3.5029</td>
<td>0.0021</td>
<td>Rej</td>
</tr>
<tr>
<td>STM</td>
<td>-0.0043</td>
<td>0.0003</td>
<td>-14.4257</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0037</td>
<td>0.0003</td>
<td>-14.5119</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Tech</td>
<td>-0.0033</td>
<td>0.0003</td>
<td>-11.1939</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0019</td>
<td>0.0002</td>
<td>-7.6263</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>Tot</td>
<td>0.0005</td>
<td>0.0002</td>
<td>2.1565</td>
<td>0.0428</td>
<td>Rej</td>
<td>-0.0003</td>
<td>0.0002</td>
<td>-1.8239</td>
<td>0.0824</td>
<td>Rej</td>
</tr>
<tr>
<td>Vall</td>
<td>-0.0028</td>
<td>0.0004</td>
<td>-6.9810</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0023</td>
<td>0.0003</td>
<td>-6.8510</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
<tr>
<td>VeoE</td>
<td>0.0004</td>
<td>0.0004</td>
<td>0.9626</td>
<td>0.3467</td>
<td>Rej</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.3993</td>
<td>0.6937</td>
<td>Rej</td>
</tr>
<tr>
<td>Vin</td>
<td>0.0008</td>
<td>0.0002</td>
<td>4.0046</td>
<td>0.0006</td>
<td>Rej</td>
<td>0.0004</td>
<td>0.0002</td>
<td>2.6432</td>
<td>0.0152</td>
<td>Rej</td>
</tr>
<tr>
<td>Viv</td>
<td>-0.0015</td>
<td>0.0001</td>
<td>-10.9753</td>
<td>0.0000</td>
<td>Rej</td>
<td>-0.0012</td>
<td>0.0001</td>
<td>-11.2501</td>
<td>0.0000</td>
<td>Rej</td>
</tr>
</tbody>
</table>

Table 9: Results of the testing the nullity hypothesis of \( \gamma_1 \) based on weekly returns with beta estimated by OLS and FLS.
References


