Creating Rigorous Mathematical Thinking: A Dynamic that Drives Mathematics and Science Conceptual Development

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Introduction

Several longitudinal studies are being conducted to demonstrate the efficacy of a new paradigm for accelerating and deepening the creation of higher-order mathematical thinking and mathematics and science conceptual development. The paradigm operationalizes constructs of a theory of rigorous mathematical thinking (Kinard, 2000) through Feuerstein's Instrumental Enrichment (FIE) program with Mediated Learning Experience (MLE, Feuerstein, 1980).

This paper presents the paradigm and some initial results from one of the studies that targets inner-city youths who have experienced previous academic failure and possess the so-called traits that are presumed to place limits on individual difference (see, for example, Hernstein and Murray, *The Bell Curve*, 1994).

The Mathematical Thinking Dynamic

Kinard (2000) defines rigorous mathematical thinking as the synthesis and utilization of mental operations to:

- derive insights about patterns and relationships;
- apply culturally derived devices and schemes to further elaborate these insights for their organization, correlation, orchestration and abstract representation to form emerging conceptualizations and understandings;
- transform and generalize these emerging conceptualizations and understandings into coherent, logically-bound ideas and networks of ideas;
- engineer the use of these ideas to facilitate problem-solving and the derivations of other novel insights in various contexts and fields of human activity; and,
- perform critical examination, analysis, introspection, and ongoing monitoring of the structures, operations, and processes of rigorous mathematical thinking for its radical self-understanding and its own intrinsic integrity.

Theoretical Construct I

A construct of this theory is that rigorous mathematical thinking is a dynamic that structures a logical framework and an organizing propensity for numerous socio-cultural endeavors through its discovery, definition, and orchestration of those qualitative and quantitative aspects of objects and events in nature and human activity. The enigma of the apparent universal intrinsic pervasiveness of order, structure, and change is continuously intriguing. It is through mathematical thinking that the human mind can attempt to discover and characterize underlying order in the face of chaos; structure in the midst of fragmentation, isolation, and incoherency; and, dynamic change in the context of constancy and steady-state behavior. Mathematical thinking structures and creatively manipulates growing systems of thought as change, order, and structure are defined and uniquely moved through a process of conceptualizing to depict and understand evident and underlying patterns and relationships for each situation under examination.

Mathematics is the study of patterns and relationships. In modern mathematics, such study is facilitated by culturally derived devices and schemes that were constructed through and are driven by the mathematical thinking dynamic. These culturally derived devices and schemes are synonymous with Vygotsky's conceptualization of psychological "tools" (see Kozulin, *Psychological Tools*, 1998). Kozulin, in elaborating on Vygotsky's conceptualization, stated, "Psychological tools are symbolic artifacts – signs, symbols, texts, formulae, graphic-symbolic devices – that help us master our own 'natural' psychological functions of perception, memory, attention, will, etc." (Kozulin, 1998).

Symbolic devices and schemes that have been developed through socio-cultural needs to facilitate mental activity dealing with patterns and relationships are mathematical psychological tools. The structuring of these tools has slowly evolved over periods of time through collective, generalized purposes of the transitioning needs of the transforming cultures (see, for example, Eves, *An Introduction to the History of Mathematics*). Both the creation of such tools and their utilization develop, solicit, and further elaborate higher-order mental processing that characterizes the mathematical thinking dynamic (see Figure 1). Mathematical psychological tools range from simple forms of symbolization such as numbers and signs in arithmetic to the complex notations and symbolizations that appear in calculus and mathematical physics such as differential equations, integral functions or Laplace Transforms. Mental operations that are synthesized, orchestrated and applied which characterize mathematical thinking are presented in Table 1. Evidence of the logical framework and organization of modern mathematics is reflected through both the hierarchal nature of its system of psychological tools and sub-disciplines and the progressive embodiment of the conceptualization process from simple arithmetic through mathematical physics.

Mathematics, with its system of psychological tools and mathematical thinking dynamic, is the primary language for basic and applied science. Language provides the vehicle for the formulation, organization, and articulation of human thought. Science is a way of knowing – a process of investigating, observing, thinking, experimenting, and validating. This way of knowing is the application of human intelligence to produce interconnected and validated ideas about how the physical, biological, psychological, and social worlds work (American Association for the Advancement of Science, 1993). Scientific thought processes comprise cognitive functions, mental operations, and emerging conceptualizations associated with this way of knowing to understand the world around us. The psychological tools of mathematics and the mathematical thinking dynamic provide the vehicle and energizing element to promote the processes of representation, synthesis and articulation – a language for scientific thought at the receptive, expressive, and elaborational levels. The American Association for the Advancement of Science states in *Science for all Americans* (1990) that “mathematics provides the grammar of science – the rules for analyzing scientific ideas and data rigorously.”

TABLE 1

Mental Operations that Characterize Mathematical Thinking

Abstract relational thinking
Structural analysis
Operational analysis
Representation
Projection of visual relationships
Inferential-hypothetical thinking
Deduction
Induction
Differentiation
Integration
Reflective thinking with elaboration of cognitive categories
Conservation of constancy in the context of dynamic change

Since mathematical thinking synthesizes and utilizes a spectrum of cognitive processing that advances onto higher and higher levels of abstraction, it has to be rigorous by its very nature. Kinard and Falik (1999) delineate the following as elements of rigor:

Fundamental Elements of Rigor

- Sharpness in focus and perception
- Clarity and completeness in definition, conceptualization, and delineation of critical attributes
- Precision and accuracy

Systemic Elements of Rigor

- Critical inquiry and intense searching for truth (logical evidence of reality)
- Intensive and aggressive mental engagement that dynamically seeks to create and sustain a higher quality of thought

Higher-order Superstructures of Rigor

- A mindset for critical engagement
- A state of vigilance that is driven by a strong, persistent, and inflexible desire to know and deeply understand

The high level of abstraction, logical integrity, and organizing propensity of mathematical thinking imbue it with an overarching usefulness and applicability that pervades and drives numerous fields of human endeavors including natural and social sciences, agriculture, art, business, engineering, history, industry, medicine, music, politics, sports, etc. The dependency of science on mathematical thinking was voiced by Plato around 390 B.C.:

> "…that the reality which scientific thought is seeking must be expressible in mathematical terms, mathematics being the most precise and definite kind of thinking of which we are capable. The significance of this idea for the development of science from the first beginnings to the present day has been immense."

Theoretical Construct II

Rigorous mathematical thinking engineers and formulates higher-order conceptual tools that produce scientific thinking and scientific conceptual development.

Theoretical Construct III

The constructs of the theory are operationalized through a paradigm that consists of MLE and FIE, along with a unique blend of the operational concept of rigorous thinking (Kinard and Falik, 1999), the appropriation of culturally derived psychological tools as described by Kozulin (1998), and Ben-Hur's model of concept development (1999).

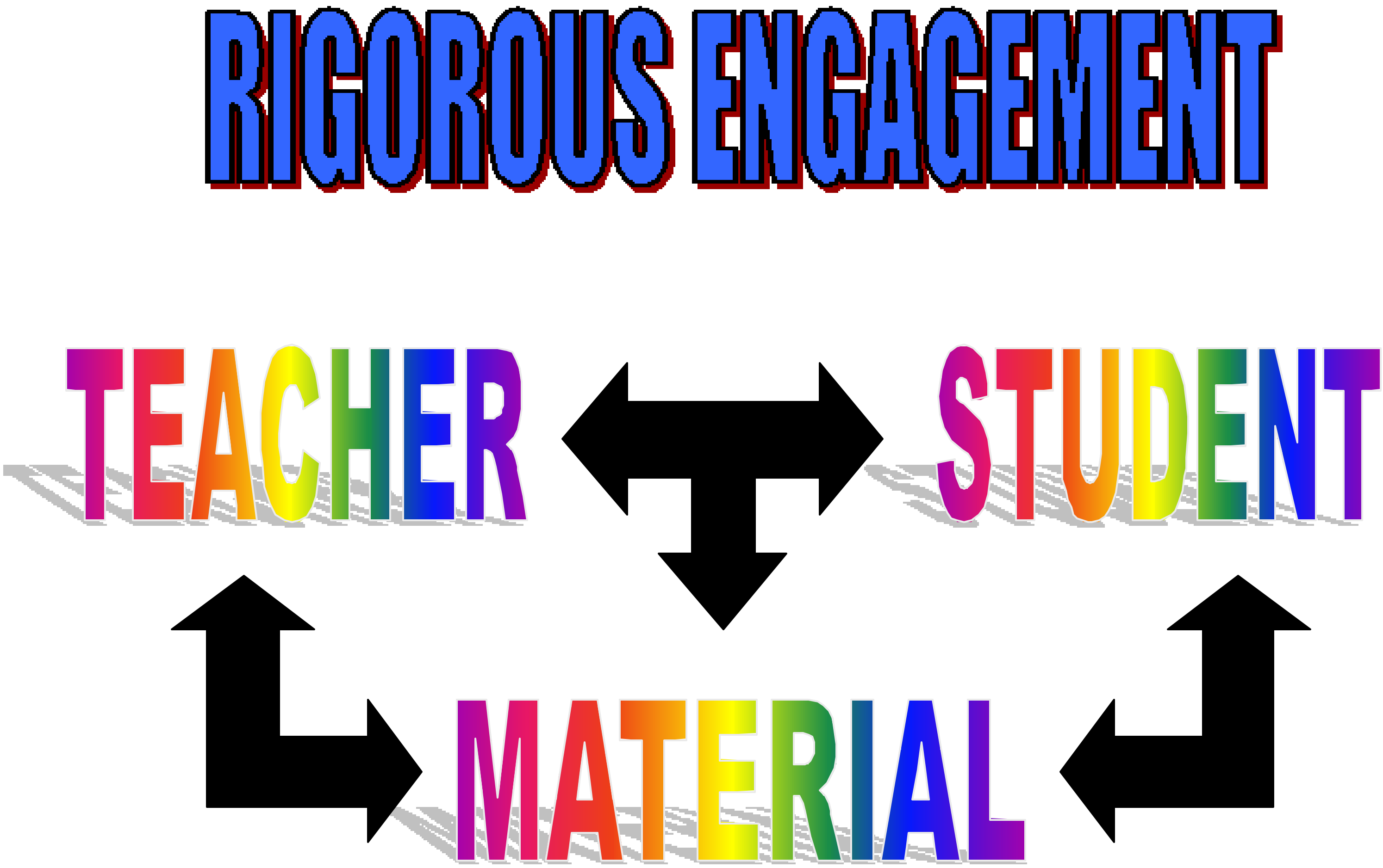
The Paradigm

Creation of rigorous mathematical thinking and mathematical-scientific conceptual development is structured and realized through rigorous engagements with patterns and relationships (see Figure 2). The structuring and maintenance of the engagement are engineered through MLE. Professor Reuven Feuerstein defines MLE as a quality or modality of learning that requires a human mediator who guides and nurtures the mediatee (learner) using three central criteria (intentionality/reciprocity, transcendence, and meaning) and other criteria that are situational (Feuerstein and Feuerstein, 1991). The learner is mediated while utilizing the comprehensive and highly systematic sets of psychological tools of the FIE program to begin realizing the six subgoals of the program: correction of deficient cognitive functions; acquisition of basic concepts, labels, vocabulary, operations, and concepts necessary for FIE; production of intrinsic motivation through habit formation; creation of task-intrinsic motivation; and, transformation of the learner's role into one of an active generator of new information.

During the realization of these subgoals many of the psychological tools of the FIE program are appropriated as mathematical psychological tools, as delineated by Kozulin (1998), using the MLE central criteria. As the learner acquires and utilizes these mathematical psychological tools to generate, transform, represent, manipulate, and apply insights derived from patterns and relationships, rigorous mathematical thinking is created. As mathematical thinking is unfolding, the learner is rigorously mediated to utilize his/her day-to-day perceptions and spontaneous concepts to construct mathematical concepts. During the process the learner is mediated to utilize his/her mathematical thinking and conceptualizing to formulate scientific conceptual tools to build higher-order scientific thinking and science concepts.

The FIE program provides rich avenues through which concept development can emerge within the learner according to the five principles of mediation practice described by Ben-Hur (1999). These five principles are: practice, both in terms of quantity and quality; decontextualization; meaning; recontextualization; and, realization.

FIGURE 2



External / Internal Environments of Student and Teacher; Lesson / Content

The interactions developed through rigor are dynamic (exciting, challenging, and invigorating), interdependent, and transformative. When these bidirectional interactions permeate each other to produce dynamic reversibility throughout the channels of interaction, rigorous engagement has been initiated.

Developed by James T. Kinard, Ph.D.

Research Results

Data were produced through pre- and post-cognitive testing, analysis of audio and video taped sessions of the interventions, case studies of students through their journals of reflection, and "talk out loud about your thinking" by students as they performed tasks and solved problems.

Pre- Post-tests in a Logico-verbal Modality

Logical Reasoning-Inference Test, RL-3

Parallel pre- and post-versions of Logical Reasoning-Inference Test, RL-3, developed by Educational Testing Service (Ekstrom, et al., 1976), were administered for each intervention. Each item on the test requires the student to read one or two statements that might appear in a newspaper or popular magazine. The student must choose only one of five statements that represents the most correct conclusion that can be drawn. The student is instructed not to consider information that is not given in the initial statement(s) to draw the most correct conclusion. The student is also advised not to guess, unless he or she can eliminate possible answers to improve the chance of choosing, since incorrectly chosen responses will count against him/her.

Ekstrom et al. (1976 and 1979) defined the cognitive factor involved in this test as "The ability to reason from premise to conclusion, or to evaluate the correctness of a conclusion." These authors further stated: "Guilford and Cattell (1971) have sometimes called this factor "Logical Evaluation."

Guilford and Hoepfner (1971) pointed out that what is called for in syllogistic reasoning tasks is not deduction but the ability to evaluate the correctness of the answers presented. This factor can be confounded with verbal reasoning when the level of reading comprehension required is not minimized.

The complexity of this factor has been pointed out by Carroll (1974) who describes it as involving both the retrieval of meanings and of algorithms from long-term memory and then performing serial operations on the materials retrieved. He feels that individual differences on this factor can be related not only to the content and temporal aspects of these operations, but also to the attention which the subject gives to details of the stimulus materials."

Three FIE-MLE practitioners, first independently and then collectively, analyzed test items on RL-3 for their required use of cognitive functions and operations to be performed successfully by the student.

The following is a summary of their work.

> The student must engage in logical reasoning which requires abstract relational thinking at various levels of complexity. The student is required to interrelate data from the statement(s) with data from potential conclusions to ensure total coherency – that is to conserve constancy in relationships and meaning at various levels of complexity and abstraction. The linkage between the sources of information (the statement(s) and the potential conclusion) is established or

denied through inferential thinking – a bridge that requires abstract relational hypothetical thinking to construct, with the underlying supports of precision and accuracy. The statement(s) and the conclusion are in a specifics-to-general or general-to-specifics relationship. The student's thinking must conserve relationships and meaning as it transforms their expressions into higher levels of abstraction in order to encompass broader spectra of abstraction and complexity and vice versa.

The primary cognitive operation required throughout each version of the test is abstract inferential relational thinking with various levels of complexity. This operation's required deductive and/or inductive thinking is created while the student draws from his/her repertoire of prior knowledge to do further relational thinking to provide the logical evidence for the evaluation of the validity of the conclusion. The range of the cognitive functions and operations for the pre-test was comparable with the range for the post-test, although not sequenced item by item.

The test is indeed in a logico-verbal modality with a demand in language use and an embedded requirement of reading comprehension at various levels of abstraction and complexity.

Pre- and Post-tests in a Figural Modality

Visualization Test - VZ-2

Parallel pre- and post-versions of Visualization Test, VZ-2, developed by Educational Testing Service (Ekstrom, et al., 1976), were administered. The authors of the test define the cognitive factor as "the ability to manipulate or transform the image of spatial patterns into other arrangements."

The instrument used in this research is the Paper Folding Test – VZ-2. The student is instructed to imagine the folding of a square piece of paper according to figures drawn to the left of a vertical line with one or two small circles drawn on the last figure to indicate where the paper has been punched through all thicknesses. The student is to decide which of five figures to the right of the vertical line will be the square sheet of paper when it is completely unfolded with a hole or holes in it. The student is admonished not to guess, since a fraction of the number incorrectly chosen will be subtracted from the number marked correctly.

Two FIE-MLE practitioners analyzed each item to determine the cognitive processing required to successfully perform the task and choose the correct answer. A summary of their findings is given below.

The student must integrate the use of relevant cues and the sequencing of figures to mentally define and restructure the components of the field onto a unified spatial presentation through visualization. There has to be a high level of conservation of constancy in size, shape, orientation, and location in the face of spatial and temporal transitions. The output requires projection of virtual relationships with precision and accuracy. Both the pre- and post-test increase, to the same degree, in difficulty from the first to the last item. The latter items require intensity in conserving constancy with very high levels of novelty, complexity, and abstraction. These items require deep internalization, integration, and structural and operational analyses.

Data for RL-3 and VZ-2 are presented in Table 2 and Figure 3. The pre-tests were administered prior to the initiation of the intervention. The post-tests were administered at 25 hours of intervention. Notice that the gain scores were positive for most students on both tests. These results demonstrate that cognitive dysfunctioning is being corrected and the mental operations of abstract relational thinking, inferential-hypothetical thinking, induction, deduction, integration, structural analysis and operational analysis are being developed. These mental operations help to characterize the mathematical thinking dynamic.

Emerging Conceptualizations and Mental Operations

A concept and mental operation that is highly fundamental to mathematical thinking is conservation of constancy in the context of dynamic change. The development of this concept and mental operation was initiated from the first sheet of the first instrument, Organization of Dots, of the FIE program.

The paradigm structures practice for the learner to develop and utilize this concept and operation in the defining, characterizing, transforming, and applying aspects of patterns and relationships through pictorial, figural, numerical, graphical-symbolic, verbal, and logical-verbal modalities. The learner must experience the emerging of each mental operation and each concept through the same rigorous protocol cited above.

A big idea that is being developed in this project is the nature and types of mathematical functions. Supporting concepts that are being mediated as emerging foundational elements to mathematical functions are: dependent and independent variables; interdependency; relations; patterns; functional relationships; rate; recursion, etc. This paradigm addresses all of the algebra standard for grades 9-12 along with expectations recommended by the National Council of Teachers of Mathematics (2000, see Table 3).

TABLE 3

Algebra STANDARD for Grades 9–12

Instructional programs from prekindergarten through grade 12 should enable all students to—

Expectations

In grades 9–12 all students should–

Understand patterns, relations, and functions	• generalize patterns using explicitly defined and recursively defined functions; • understand relations and functions and select, convert flexibly among, and use various representations for them; • analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior; • understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions; • understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions; • interpret representations of functions of two variables.
Represent and analyze mathematical situations and structures using algebraic symbols	• understand the meaning of equivalent forms of expressions, equations, inequalities, and relations; • write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency–mentally or with paper and pencil in simple cases and using technology in all cases; • use symbolic algebra to represent and explain mathematical relationships; • use a variety of symbolic representations, including recursive and parametric equations, for functions and relations; • judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
Use mathematical models to represent and understand quantitative relationships	• identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships; • use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts; • draw reasonable conclusions about a situation being modeled.
Analyze change in various contexts	• approximate and interpret rates of change from graphical and numerical data.

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The concept of a mathematical function began to emerge when students began to verbalize their insights. The following is a sampling of these insights.

Student Insights

Student #1: "So when we look back at page 1 of Organization of Dots, the cultural attributes of a square are in a functional relationship with each other to form the square."

Student #2: "Each characteristic of the square, then, is an independent variable."

Mediator: "Is there another type of variable?"

Student #2: "Yes, the dependent variable, the square itself. The square is a function of its parts and their relationships."

Student #1: "There is another point now that we are going beneath the surface, trying to go deeper. Sides of the square – the opposite sides are parallel to each other. If I am standing in the center of the square I will be in a lot of parallelism. Where did it come from? The opposite sides. The parallelism is a dependent variable. It depends on the equidistance of the opposite sides. It is a function of these independent variables. There are two functions embedded here – the square and the parallelism."

At the writing of this paper, the psychological tools of four FIE instruments had been appropriated and were utilized by students to create mathematical thinking. The four instruments are: Organization of Dots and Orientation in Space I, Adult Version, Analytic Perception, and Numerical Progressions. The concept of mathematical function with independent and dependent variables was experienced through most pages and through all modalities. Students are beginning to represent higher-order functional relationships – linear, quadratic, and exponential functions – and manipulate them within the rules of logic and relate them in terms of expressing various empirical and scientific realities. They are using mathematical thinking to characterize, quantify, and further understand growth, decay, surface areas and changes in surface areas of, for example, a cube of melting ice, molecular motion, etc. Many are becoming fluid in articulating their thinking through reflection and elaboration of cognitive categories.

At this point, 85% of the students are developing a profound love for doing rigorous mathematical thinking. Secondly, most students demonstrate task-intrinsic motivation and a competitive spirit when doing inductive thinking to construct generalizations. When one student was mediating the class to understand why his plan of action worked to perform a task that required mathematical thinking, he said, "use your mental operations to play with the options. Enjoy using your mental processes to create different strategies. Have fun organizing and reorganizing your cognitive functions and operations as you work through the problem."

Examples of students' work are presented below and in Figures 4 and 5.

Just prior to the writing of this paper, students were asked to write their perceptions of mathematical thinking based on their experiences in the class. This is a collection of some of their responses.

"When you have to synthesize, develop, direct, orchestrate Mental Operations that have inside of them Cognitive Functions. A concept of using Mathematical Terms to solve everyday problems in life. Identify and visualizing at all times. A conscious awareness of issues, complications and processes where you precisely connect the proper mental operations to the issue or equation."

"Mathematical Thinking: The construction of mental operations to gain in site about a pattern or relationship and represent them by symbols."

"Mathematical thinking is a serious engagement in developing an analytic perception at all times. It also is a mental operation that helps you gain insight about patterns and relationships."

"Mathematical thinking is a conscious awareness of issues, complication and processes where you precisely connect the proper mental operations through analytical perception to illustrate the correct answer to the issue or equation. Get the construction of mental operation."

"Mathematical Thinking is a process using your cognitive functions and sociological tools to apply and figure out tasks that relate to everyday situations as well as equations."

"Mathematical thinking is the conscious act of relating comparing and finding patterns and sequences of events through numerical symbology, everything has a number. Therefore there must be some law or order underlying it all which can be made into an equation every time to benefit our mental and physical states."

"Mathematical thinking: In definition, it is similar to an injustice to the concept. Many thoughts come to mind since correlation as we know it is based on mathematical thinking. For example, natural life processes pack, which causes life in a result of mathematical thinking in animate action. The specifics of this process show you how the structure of your inspiratory system and it parts work together in a systematic sequential pattern for you to function. This begins to start cycles which allows one to experience more and develop higher orders of mathematical thinking as one lives."

"Mathematical Thinking is a group of cognitive functions used to prove thought fundamental and all life related situation deal with laws and actual facts."

FIGURE 4

A Sample of a student's work when doing higher-order mathematical thinking: Developing and transforming insights about relationships between relationships and mathematical functions. Note: This work was produced spontaneously by the student when working on a task far remote to it. It is only though deep structural thinking that such transcendence could be made.

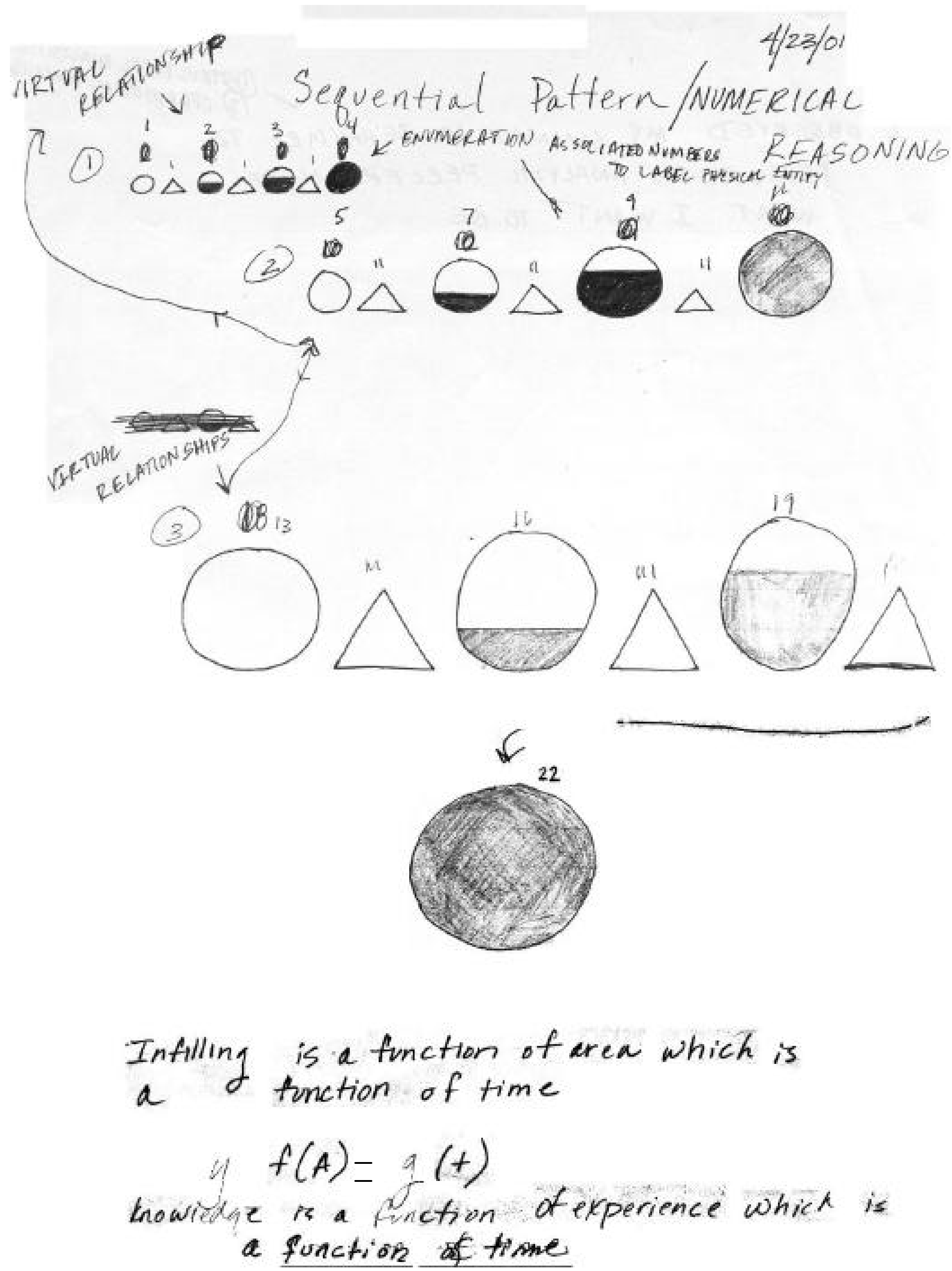
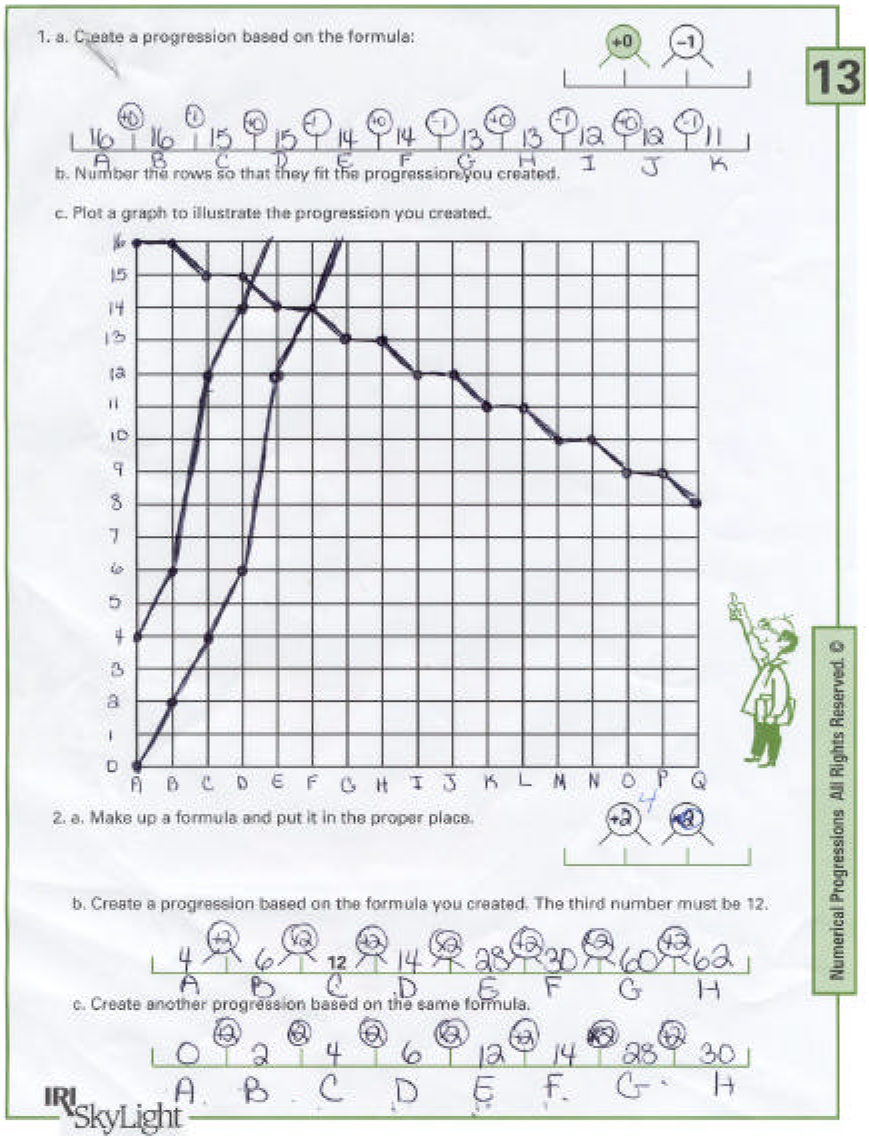


FIGURE 5

Example Of a Student's work showing how he is using mathematical thinking to traverse modalities (Numerical, Graphical, Logical- verbal) as he does deduction and induction.



"I was <u>relating</u> the graph, with its horizontal and vertical axis, coordinates and numerical modalities, to a company on the stock market's (Ex. 2-C on Graph) growth within the first 17 months (graph represents profit in $10,000's and also times passed, months). In the first month, you have nothing, you borrow from banks, promoting your product, trying to get investors to invest in your stock, Gain is <u>Break Even</u> to <u>Minimum profit.</u> (A,O) In the second month you make 20,000 profit, and the third and the forth. What we barely realize is that 100% profit is being made in each and every month, though $20,000 profit seems little at the time. But as you have more money to invest, your profit will also, in this case be better."

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